

General Pontryagin-Type Stochastic Maximum Principle and Backward Stochastic Evolution Equations in Infinite Dimensions

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Abstract

The main purpose of this paper is to give a solution to a long-standing unsolved problem in stochastic control theory, i.e., to establish the Pontryagin-type maximum principle for optimal controls of general infinite dimensional nonlinear stochastic evolution equations. Both drift and diffusion terms can contain the control variables, and the control domains are allowed to be nonconvex. The key to reach it is to provide a suitable formulation of operator-valued backward stochastic evolution equations (BSEEs for short), as well as a way to define their solutions. Besides, both vector-valued and operator-valued BSEEs, with solutions in the sense of transposition, are studied. As a crucial preliminary, some weakly sequential Banach-Alaoglu-type theorems are established for uniformly bounded linear operators between Banach spaces.

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Key Words. Stochastic evolution equation, optimal control, Pontryagin-type maximum principle, backward stochastic evolution equation, transposition solution.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$, on which a one-dimensional standard Brownian motion $\{w(t)\}_{t \geq 0}$ is defined. Let $T > 0$, and let X be a Banach space. For any $t \in [0, T]$ and $r \in [1, \infty)$, denote by $L^r_{\mathcal{F}_t}(\Omega; X)$ the Banach space of all \mathcal{F}_t -measurable random variables $\xi : \Omega \rightarrow X$ such that $\mathbb{E}|\xi|_X^r < \infty$, with the canonical norm. Also, denote by $D_{\mathbb{F}}([0, T]; L^r(\Omega; X))$ the vector space of all X -valued r th power integrable \mathbb{F} -adapted processes $\phi(\cdot)$ such that $\phi(\cdot) : [0, T] \rightarrow L^r(\Omega, \mathcal{F}_T, P; X)$ is càdlàg, i.e., right continuous with left limits. Clearly, $D_{\mathbb{F}}([0, T]; L^r(\Omega; X))$ is a Banach space with the following norm

$$|\phi(\cdot)|_{D_{\mathbb{F}}([0, T]; L^r(\Omega; X))} = \sup_{t \in [0, T]} [\mathbb{E}|\phi(t)|_X^r]^{1/r}.$$

We denote by $C_{\mathbb{F}}([0, T]; L^r(\Omega; X))$ the Banach space of all X -valued \mathbb{F} -adapted processes $\phi(\cdot)$ such that $\phi(\cdot) : [0, T] \rightarrow L^r(\Omega, \mathcal{F}_T, P; X)$ is continuous, with the norm inherited from $D_{\mathbb{F}}([0, T]; L^r(\Omega; X))$. Fix any $r_1, r_2, r_3, r_4 \in [1, \infty]$. Put

$$\begin{aligned} L_{\mathbb{F}}^{r_1}(\Omega; L^{r_2}(0, T; X)) &= \left\{ \varphi : (0, T) \times \Omega \rightarrow X \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \mathbb{E} \left(\int_0^T |\varphi(t)|_X^{r_2} dt \right)^{\frac{r_1}{r_2}} < \infty \right\}, \\ L_{\mathbb{F}}^{r_2}(0, T; L^{r_1}(\Omega; X)) &= \left\{ \varphi : (0, T) \times \Omega \rightarrow X \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted and } \int_0^T \left(\mathbb{E}|\varphi(t)|_X^{r_1} \right)^{\frac{r_2}{r_1}} dt < \infty \right\}. \end{aligned}$$

Clearly, both $L_{\mathbb{F}}^{r_1}(\Omega; L^{r_2}(0, T; X))$ and $L_{\mathbb{F}}^{r_2}(0, T; L^{r_1}(\Omega; X))$ are Banach spaces with the canonical norms. If $r_1 = r_2$, we simply denote the above space by $L_{\mathbb{F}}^{r_1}(0, T; X)$. Let Y be another Banach space. Denote by $\mathcal{L}(X, Y)$ the (Banach) space of all bounded linear operators from X to Y , with the usual operator norm (When $Y = X$, we simply write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, Y)$). Further, we denote by $\mathcal{L}_{pd}(L_{\mathbb{F}}^{r_1}(0, T; L^{r_2}(\Omega; X)), L_{\mathbb{F}}^{r_3}(0, T; L^{r_4}(\Omega; Y)))$ (*resp.* $\mathcal{L}_{pd}(X, L_{\mathbb{F}}^{r_3}(0, T; L^{r_4}(\Omega; Y)))$) the vector space of all bounded, pointwisely defined linear operators \mathcal{L} from $L_{\mathbb{F}}^{r_1}(0, T; L^{r_2}(\Omega; X))$ (*resp.* X) to $L_{\mathbb{F}}^{r_3}(0, T; L^{r_4}(\Omega; Y))$, i.e., for a.e. $(t, \omega) \in (0, T) \times \Omega$, there exists an $L(t, \omega) \in \mathcal{L}(X, Y)$ verifying that $(\mathcal{L}u(\cdot))(t, \omega) = L(t, \omega)u(t, \omega)$, $\forall u(\cdot) \in L_{\mathbb{F}}^{r_1}(0, T; L^{r_2}(\Omega; X))$ (*resp.* $(\mathcal{L}x)(t, \omega) = L(t, \omega)x$, $\forall x \in X$). Similarly, one can define the spaces $\mathcal{L}_{pd}(L^{r_2}(\Omega; X), L_{\mathbb{F}}^{r_3}(0, T; L^{r_4}(\Omega; Y)))$ and $\mathcal{L}_{pd}(L^{r_2}(\Omega; X), L^{r_4}(\Omega; Y))$, etc.

Let H be a complex Hilbert space, and let A be an unbounded linear operator (with domain $D(A)$ on H), which is the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t \geq 0}$. Denote by A^* the dual operator of A . Clearly, $D(A)$ is a Hilbert space with the usual graph norm, and A^* is the infinitesimal generator of $\{S^*(t)\}_{t \geq 0}$, the dual C_0 -semigroup of $\{S(t)\}_{t \geq 0}$. For any $\lambda \in \rho(A)$, the resolvent of A , denote by A_λ the Yosida approximation of A and by $\{S_\lambda(t)\}_{t \in \mathbb{R}}$ the C_0 -group generated by A_λ . Let U be a metric space with its metric $d(\cdot, \cdot)$. Put

$$\mathcal{U}[0, T] \triangleq \left\{ u(\cdot) : [0, T] \rightarrow U \mid u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right\}.$$

Throughout this paper, we assume the following condition.

(A1) Suppose that $a(\cdot, \cdot, \cdot) : [0, T] \times H \times U \rightarrow H$ and $b(\cdot, \cdot, \cdot) : [0, T] \times H \times U \rightarrow H$ are two maps satisfying: i) For any $(x, u) \in H \times U$, the maps $a(\cdot, x, u) : [0, T] \rightarrow H$ and $b(\cdot, x, u) : [0, T] \rightarrow H$ are Lebesgue measurable; ii) For any $(t, x) \in [0, T] \times H$, the maps $a(t, x, \cdot) : U \rightarrow H$ and $b(t, x, \cdot) : U \rightarrow H$ are continuous; and iii) There is a constant $C_L > 0$ such that

$$\begin{cases} |a(t, x_1, u) - a(t, x_2, u)|_H + |b(t, x_1, u) - b(t, x_2, u)|_H \leq C_L |x_1 - x_2|_H, \\ |a(t, 0, u)|_H + |b(t, 0, u)|_H \leq C_L, \end{cases} \quad \forall (t, x_1, x_2, u) \in [0, T] \times H \times H \times U. \quad (1.1)$$

Consider the following controlled (forward) stochastic evolution equation:

$$\begin{cases} dx = [Ax + a(t, x, u)]dt + b(t, x, u)dw(t) & \text{in } (0, T], \\ x(0) = x_0, \end{cases} \quad (1.2)$$

where $u \in \mathcal{U}[0, T]$ and $x_0 \in L_{\mathcal{F}_0}^{p_0}(\Omega; H)$ for some given $p_0 > 1$. We call $x(\cdot) \equiv x(\cdot; x_0, u) \in C_{\mathbb{F}}([0, T]; L^{p_0}(\Omega; H))$ a mild solution to (1.2) if

$$x(t) = S(t)x_0 + \int_0^t S(t-s)a(s, x(s), u(s))ds + \int_0^t S(t-s)b(s, x(s), u(s))dw(s), \quad \forall t \in [0, T].$$

In the sequel, we shall denote by C a generic constant, depending on T, A, p_0 (or p to be introduced later) and C_L (or J and K to be introduced later), which may be different from one place to another. Similar to [9, Chapter 7], it is easy to show the following result:

Lemma 1.1 *Let the assumption (A1) hold. Then, the equation (1.2) is well-posed in the sense of mild solution. Furthermore,*

$$|x(\cdot)|_{C_{\mathbb{F}}([0, T]; L^{p_0}(\Omega; H))} \leq C(1 + |x_0|_{L_{\mathcal{F}_0}^{p_0}(\Omega; H)}).$$

Also, we need the following condition:

(A2) *Suppose that $g(\cdot, \cdot, \cdot) : [0, T] \times H \times U \rightarrow \mathbb{R}$ and $h(\cdot) : H \rightarrow \mathbb{R}$ are two functions satisfying:*
i) For any $(x, u) \in H \times U$, the function $g(\cdot, x, u) : [0, T] \rightarrow \mathbb{R}$ is Lebesgue measurable; ii) For any $(t, x) \in [0, T] \times H$, the function $g(t, x, \cdot) : U \rightarrow \mathbb{R}$ is continuous; and iii) There is a constant $C_L > 0$ such that

$$\begin{cases} |g(t, x_1, u) - g(t, x_2, u)|_H + |h(x_1) - h(x_2)|_H \leq C_L|x_1 - x_2|_H, \\ |g(t, 0, u)|_H + |h(0)|_H \leq C_L, \end{cases} \quad \forall (t, x_1, x_2, u) \in [0, T] \times H \times H \times U. \quad (1.3)$$

Define a cost functional $\mathcal{J}(\cdot)$ (for the controlled system (1.2)) as follows:

$$\mathcal{J}(u(\cdot)) \triangleq \mathbb{E} \left[\int_0^T g(t, x(t), u(t))dt + h(x(T)) \right], \quad \forall u(\cdot) \in \mathcal{U}[0, T], \quad (1.4)$$

where $x(\cdot)$ is the corresponding solution to (1.2).

Let us consider the following optimal control problem for the system (1.2):

Problem (P) *Find a $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ such that*

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} \mathcal{J}(u(\cdot)). \quad (1.5)$$

Any $\bar{u}(\cdot)$ satisfying (1.5) is called an optimal control. The corresponding state process $\bar{x}(\cdot)$ is called an optimal state (process), and $(\bar{x}(\cdot), \bar{u}(\cdot))$ is called an optimal pair.

The main goal of this paper is to establish some necessary conditions for optimal pairs of Problem (P), in the spirit of the Pontryagin-type maximum principle ([25]). In this respect, the problem is now well-understood in the case that $\dim H < \infty$. We refer to [14] and the references therein for early studies on the maximum principle for controlled stochastic differential equations in finite dimensional spaces. Then, people established further results on the maximum principle for stochastic control systems under various assumptions, say, the diffusion coefficients were non-degenerate

(e.g. [11]), and/or the diffusion coefficients were independent of the controls (e.g. [4, 6]), and/or the control domains were convex (e.g. [4]). Note that, generally speaking, many practical systems (especially in the area of finance) do not satisfy these assumptions. In [24], a maximum principle was obtained for general stochastic control systems without the above mentioned assumptions, and it was found that the corresponding result in the general case differs essentially from its deterministic counterpart. As important byproducts in the study of the above finite dimensional stochastic control problems, one introduced some new mathematical tools, say, backward stochastic differential equations (BSDEs, for short) and forward-backward stochastic differential equations ([6, 7, 23] and [19, 29]), which are now extensively applied to many other fields.

Let us recall here the main idea and result in [24]. Suppose that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a given optimal pair for the special case that $A = 0$, $H = \mathbb{R}^n$ (for some $n \in \mathbb{N}$) and \mathbb{F} is the natural filtration \mathbb{W} (generated by the Brownian motion $\{w(\cdot)\}$ and augmented by all the \mathbb{P} -null sets). First, similar to the corresponding deterministic setting, one introduces the following first order adjoint equation (which is however a BSDE in the stochastic case):

$$\begin{cases} dy(t) = - \left[a_x(t, \bar{x}(t), \bar{u}(t))^\top y(t) + b_x(t, \bar{x}(t), \bar{u}(t))^\top Y(t) \right. \\ \quad \left. - g_x(t, \bar{x}(t), \bar{u}(t)) \right] dt + Y(t) dw(t), & \text{in } [0, T], \\ y(T) = -h_x(\bar{x}(T)). \end{cases} \quad (1.6)$$

Here the unknown is a *pair* of \mathbb{F} -adapted processes $(y(\cdot), Y(\cdot)) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; \mathbb{R}^n)) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^n)$. Next, to establish the desired maximum principle for stochastic controlled systems with control-dependent diffusion and possibly nonconvex control domains, the author in [24] had the following fundamental finding: Except for the first order adjoint equation (1.6), one has to introduce an additional second order adjoint equation as follows:

$$\begin{cases} dP(t) = - \left[a_x(t, \bar{x}(t), \bar{u}(t))^\top P(t) + P(t) a_x(t, \bar{x}(t), \bar{u}(t)) \right. \\ \quad + b_x(t, \bar{x}(t), \bar{u}(t))^\top P(t) b_x(t, \bar{x}(t), \bar{u}(t)) \\ \quad + b_x(t, \bar{x}(t), \bar{u}(t))^\top Q(t) + Q(t) b_x(t, \bar{x}(t), \bar{u}(t)) \\ \quad \left. + \mathbb{H}_{xx}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t)) \right] dt + Q(t) dw(t), & \text{in } [0, T] \\ P(T) = -h_{xx}(\bar{x}(T)). \end{cases} \quad (1.7)$$

In (1.7), the *Hamiltonian* $\mathbb{H}(\cdot, \cdot, \cdot, \cdot, \cdot)$ is defined by

$$\begin{aligned} \mathbb{H}(t, x, u, y_1, y_2) &= \langle y_1, a(t, x, u) \rangle_{\mathbb{R}^n} + \langle y_2, b(t, x, u) \rangle_{\mathbb{R}^n} - g(t, x, u), \\ (t, x, u, y_1, y_2) &\in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n. \end{aligned}$$

Clearly, the equation (1.7) is an $\mathbb{R}^{n \times n}$ -valued BSDE in which the unknown is a pair of processes $(P(\cdot), Q(\cdot)) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; \mathbb{R}^{n \times n})) \times L_{\mathbb{F}}^2(0, T; \mathbb{R}^{n \times n})$. Then, associated with the 6-tuple $(\bar{x}(\cdot), \bar{u}(\cdot), y(\cdot), Y(\cdot), P(\cdot), Q(\cdot))$, define

$$\begin{aligned} \mathcal{H}(t, x, u) &\triangleq \mathbb{H}(t, x, u, y(t), Y(t)) + \frac{1}{2} \langle P(t) b(t, x, u), b(t, x, u) \rangle_{\mathbb{R}^n} \\ &\quad - \langle P(t) b(t, \bar{x}(t), \bar{u}(t)), b(t, x, u) \rangle_{\mathbb{R}^n}. \end{aligned}$$

The main result in [24] asserts that the optimal pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ verifies the following stochastic maximum principle:

$$\mathcal{H}(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} \mathcal{H}(t, \bar{x}(t), u), \quad \text{a.e. } t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

On the other hand, there exist extensive works addressing the Pontryagin-type maximum principle for optimal controls of deterministic infinite dimensional controlled systems (e.g. [15] and the rich references therein). Naturally, one expects to extend the optimal control theory of both stochastic finite dimensional systems and deterministic infinite dimensional systems to that of infinite dimensional stochastic evolution equations. In this respect, we refer to [5] for a pioneer work. Later progresses are available in the literature [2, 3, 12, 26, 27, 30] and so on. Nevertheless, almost all of the existing published works on the necessary conditions for optimal controls of infinite dimensional stochastic evolution equations addressed only the case that the diffusion term does NOT depend on the control variable (i.e., the function $b(t, x, u)$ in (1.2) is independent of u). As far as we know, the stochastic maximum principle for general infinite dimensional nonlinear stochastic systems with control-dependent diffusion coefficients and possibly nonconvex control domains has been a longstanding unsolved problem.

In this paper, we aim to give a solution to the above mentioned unsolved problem. Inspired by [24], we will first study an H -valued BSEE and an $\mathcal{L}(H)$ -valued BSEE, employed accordingly as the first order adjoint equation and the second order adjoint equation (for the original equation (1.2)), and then establish the desired necessary conditions for optimal controls with the aid of the solutions of these equations.

First, we need to study the following H -valued BSEE:

$$\begin{cases} dy(t) = -A^*y(t)dt + f(t, y(t), Y(t))dt + Y(t)dw(t) & \text{in } [0, T), \\ y(T) = y_T. \end{cases} \quad (1.8)$$

Here $y_T \in L^p_{\mathcal{F}_T}(\Omega; H)$ with $p \in (1, 2]$, $f(\cdot, \cdot, \cdot) : [0, T] \times H \times H \rightarrow H$ satisfies

$$\begin{cases} f(\cdot, 0, 0) \in L^1_{\mathbb{F}}(0, T; L^p(\Omega; H)), \\ |f(t, x_1, y_1) - f(t, x_2, y_2)|_H \leq C_L(|x_1 - x_2|_H + |y_1 - y_2|_H), \\ \text{a.e. } (t, \omega) \in [0, T] \times \Omega, \quad \forall x_1, x_2, y_1, y_2 \in H. \end{cases} \quad (1.9)$$

Since neither the usual natural filtration condition nor the quasi-left continuity is assumed for the filtration \mathbb{F} in this paper, and because the unbounded operator A is assumed to generate a general C_0 -semigroup, we cannot apply the existing results on infinite dimensional BSEs (e.g. [1, 13, 19, 20]) to obtain the well-posedness of the equation (1.8).

Next, it is more important that the following $\mathcal{L}(H)$ -valued BSEE¹:

$$\begin{cases} dP = -(A^* + J^*)Pdt - P(A + J)dt - K^*PKdt - (K^*Q + QK)dt + Fdt + Qdw(t) & \text{in } [0, T), \\ P(T) = P_T \end{cases} \quad (1.10)$$

should be studied. Here and henceforth, $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))$, and $J, K \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))$. For the special case when $H = \mathbb{R}^n$, it is easy to see that (1.10) is an $\mathbb{R}^{n \times n}$ (matrix)-BSDE, and therefore, the desired well-posedness follows from that of an \mathbb{R}^{n^2} (vector)-valued BSDE. However, one has to face a real challenge in the study of (1.10) when $\dim H = \infty$, without further assumption on the data F and P_T . Indeed, in the infinite dimensional setting, although $\mathcal{L}(H)$ is still a Banach space, it is neither reflexive (needless to say to be a Hilbert space) nor separable even if H itself is separable (See Problem 99 in [10]). As far as we know, in the previous literatures there exists no such a stochastic integration/evolution equation theory in

¹Throughout this paper, for any operator-valued process (resp. random variable) R , we denote by R^* its pointwisely dual operator-valued process (resp. random variable). For example, if $R \in L^{r_1}_{\mathbb{F}}(0, T; L^{r_2}(\Omega; \mathcal{L}(H)))$, then $R^* \in L^{r_1}_{\mathbb{F}}(0, T; L^{r_2}(\Omega; \mathcal{L}(H)))$, and $|R|_{L^{r_1}_{\mathbb{F}}(0, T; L^{r_2}(\Omega; \mathcal{L}(H)))} = |R^*|_{L^{r_1}_{\mathbb{F}}(0, T; L^{r_2}(\Omega; \mathcal{L}(H)))}$.

general Banach spaces that can be employed to treat the well-posedness of (1.10). For example, the existing result on stochastic integration/evolution equation in UMD Banach spaces (e.g. [21, 22]) does not fit the present case because, if a Banach space is UMD, then it is reflexive.

The key of this work is to give accordingly reasonable definitions of the solutions to (1.8) and (1.10), and show the corresponding well-posedness results. For this purpose, we employ the transposition method developed in our previous work [18], which was addressed to the BSDEs in \mathbb{R}^n . Our method has several advantages. The first one is that the usual duality relationship is contained in our definition of solutions, and therefore, we do NOT need to use Itô's formula to derive this sort of relation as usual to obtain the desired stochastic maximum principle. Note that, it may be very difficult to derive the desired Itô's formula for the mild solutions of general stochastic evolution equations in infinite dimensions. The second one is that we do NOT need to use the Martingale Representation Theorem, and therefore we can study the problem with a general filtration. Note that, when we deal with BSEEs with operator unknowns, as far as we know, there exists no Martingale Representation Theorem (for the $\mathcal{L}(H)$ -valued martingale) even if \mathbb{F} is the natural filtration \mathbb{W} . Thirdly, as shown in [28] (though it addressed only BSDEs in \mathbb{R}^n), similar to the classical finite element method solving deterministic partial differential equations, our transposition method leads naturally numerical schemes to solve both vector-valued and operator-valued BSEEs (The detailed analysis is beyond the scope of this paper and will be presented in our forthcoming work).

In order to define the transposition solution to (1.8), we introduce the following (forward) stochastic evolution equation:

$$\begin{cases} dz = (Az + v_1)ds + v_2dw(s) & \text{in } (t, T], \\ z(t) = \eta, \end{cases} \quad (1.11)$$

where $t \in [0, T]$, $v_1 \in L^1_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $v_2 \in L^2_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$, and $q = \frac{p}{p-1}$ (See [9, Chapter 6] for the well-posedness of (1.11) in the sense of mild solution). We now introduce the following notion.

Definition 1.1 *We call $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0, T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}(0, T; L^p(\Omega; H))$ a transposition solution to (1.8) if for any $t \in [0, T]$, $v_1(\cdot) \in L^1_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $v_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^q(\Omega; H))$, $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$ and the corresponding solution $z \in C_{\mathbb{F}}([t, T]; L^q(\Omega; H))$ to the equation (1.11), it holds that*

$$\begin{aligned} & \mathbb{E}\langle z(T), y_T \rangle_H - \mathbb{E} \int_t^T \langle z(s), f(s, y(s), Y(s)) \rangle_H ds \\ &= \mathbb{E}\langle \eta, y(t) \rangle_H + \mathbb{E} \int_t^T \langle v_1(s), y(s) \rangle_H ds + \mathbb{E} \int_t^T \langle v_2(s), Y(s) \rangle_H ds. \end{aligned} \quad (1.12)$$

On the other hand, to define the solution to (1.10) in the transposition sense, we need to introduce the following two (forward) stochastic evolution equations:

$$\begin{cases} dx_1 = (A + J)x_1ds + u_1ds + Kx_1dw(s) + v_1dw(s) & \text{in } (t, T], \\ x_1(t) = \xi_1 \end{cases} \quad (1.13)$$

and

$$\begin{cases} dx_2 = (A + J)x_2ds + u_2ds + Kx_2dw(s) + v_2dw(s) & \text{in } (t, T], \\ x_2(t) = \xi_2. \end{cases} \quad (1.14)$$

Here $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1, v_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$. Also, we need to introduce the solution space for (1.10). For this purpose, write

$$\begin{aligned} & D_{\mathbb{F},w}([0, T]; L^2(\Omega; \mathcal{L}(H))) \\ & \triangleq \left\{ P(\cdot, \cdot) \mid P(\cdot, \cdot) \in \mathcal{L}_{pd}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))), \right. \\ & \quad \text{and for every } t \in [0, T] \text{ and } \xi \in L^4_{\mathcal{F}_t}(\Omega; H), \\ & \quad \left. P(\cdot, \cdot)\xi \in D_{\mathbb{F}}([t, T]; L^{\frac{4}{3}}(\Omega; H)) \text{ and } |P(\cdot, \cdot)\xi|_{D_{\mathbb{F}}([t, T]; L^{\frac{4}{3}}(\Omega; H))} \leq C|\xi|_{L^4_{\mathcal{F}_t}(\Omega; H)} \right\}, \end{aligned} \quad (1.15)$$

and

$$L^2_{\mathbb{F},w}(0, T; L^2(\Omega; \mathcal{L}(H))) \triangleq \mathcal{L}_{pd}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^1_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))). \quad (1.16)$$

We now define the transposition solution to (1.10) as follows:

Definition 1.2 We call $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F},w}([0, T]; L^2(\Omega; \mathcal{L}(H))) \times L^2_{\mathbb{F},w}(0, T; L^2(\Omega; \mathcal{L}(H)))$ a *transposition solution to the equation (1.10)* if for any $t \in [0, T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$, it holds that

$$\begin{aligned} & \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \\ & = \mathbb{E} \langle P(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s) u_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) x_1(s), u_2(s) \rangle_H ds \\ & \quad + \mathbb{E} \int_t^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) v_1(s), K(s) x_2(s) + v_2(s) \rangle_H ds \\ & \quad + \mathbb{E} \int_t^T \langle Q(s) v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q(s) x_1(s), v_2(s) \rangle_H ds. \end{aligned} \quad (1.17)$$

Here, $x_1(\cdot)$ and $x_2(\cdot)$ solve (1.13) and (1.14), respectively.

We shall derive the well-posedness of (1.8) in the sense of transposition solution, by the method developed in [18]. Here, we face to another difficulty in the study of (1.8), i.e., H is not separable in our case. On the other hand, it seems very difficult to establish the well-posedness of transposition solutions to the general equation (1.10), and therefore, in this paper we succeed in doing it only for a particular case. Because of this, instead, we introduce a weaker notion, i.e., relaxed transposition solution to (1.10) (See Definition 6.1 in Section 6). Nevertheless, it is still highly technical to derive the well-posedness result for (1.10) in the sense of relaxed transposition solution. To do this, we need to prove some weakly sequential compactness results in the spirit of the classical (sequential) Banach-Alaoglu theorem (also known as Alaoglu's theorem, e.g. [8]) but for uniformly bounded linear operators in Banach spaces. It seems that these sequential compactness results have some independent interest and may be applied in other places. Once the well-posedness for both (1.8) and (1.10), as well as some properties of the relaxed transposition solution to (1.10), are established, we are able to derive the desired Pontryagin-type stochastic maximum principle for Problem (P).

In this paper, in order to present the key idea in the simplest way, we do not pursue the full technical generality. Firstly, we consider only the simplest case of one dimensional standard Brownian motion (with respect to the time t). It would be interesting to extend the results in this paper to the case of colored (infinite dimensional) noise, or even with both time- and space-dependent noise. Secondly, we impose considerably strong regularity and boundedness assumptions on the nonlinearities appeared in the state equation (1.2) and the cost functional (1.4) (See (A1) and

(A2) in the above, (A3) in Section 8, and (A4) in Section 9). It would be quite interesting to study the same problem but with the minimal regularities and/or with unbounded controls. Thirdly, we consider neither state constraints nor partial observations in our optimal control problem.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results. Section 3 is addressed to the well-posedness of the equation (1.8). In Section 4, we study the well-posedness of the equation (1.10) under some additional assumptions. Section 5 provides some sequential Banach-Alaoglu-type theorems for uniformly bounded linear operators between Banach spaces. In Section 6, we establish the well-posedness of the equation (1.10) in the general case, while Section 7 provides further properties for solutions to this equation. Section 8 gives the Pontryagin-type necessary conditions for the optimal pair of Problem (P) under the condition that U is a convex subset in some Hilbert space. Finally, in Section 9, we establish the Pontryagin-type stochastic maximum principle for Problem (P) for the general control domain U .

2 Preliminaries

In this section, we present some preliminary results which will be used in the sequel.

First, we recall the following Burkholder-Davis-Gundy inequality in infinite dimensions (See [16, Theorem 1.2.4], for example).

Lemma 2.1 *Let $f(\cdot) \in L^2_{\mathbb{F}}(0, T; H)$. Then for any $\alpha > 0$, we have that*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left| \int_0^t f(s) dw(s) \right|_H^\alpha\right) \leq C \mathbb{E}\left(\int_0^T |f(s)|_H^2 ds\right)^{\frac{\alpha}{2}}. \quad (2.1)$$

Next, for any given $r \in [1, \infty]$, we denote by $C_{0, \mathbb{F}}^\infty((0, T); L^r(\Omega; H))$ the set of all H -valued r th power integrable \mathbb{F} -adapted processes $\phi(\cdot)$ such that $\phi(\cdot) : (0, T) \rightarrow L^r(\Omega, \mathcal{F}_T, P; H)$ is an infinitely differentiable (vector-valued) function and has a compact support in $(0, T)$. We have the following result.

Lemma 2.2 *The space $C_{0, \mathbb{F}}^\infty((0, T); L^r(\Omega; H))$ is dense in $L^s_{\mathbb{F}}(0, T; L^r(\Omega; H))$ for any $r \in [1, \infty]$ and $s \in [1, \infty)$.*

Proof: It suffices to show that for any given $f \in L^s_{\mathbb{F}}(0, T; L^r(\Omega; H))$ and each $\varepsilon > 0$, there is a $g \in C_{0, \mathbb{F}}^\infty((0, T); L^r(\Omega; H))$ such that $|f - g|_{L^s_{\mathbb{F}}(0, T; L^r(\Omega; H))} < \varepsilon$. Since the set of simple processes is dense in $L^s_{\mathbb{F}}(0, T; L^r(\Omega; H))$, we can find an

$$f_n = \sum_{i=1}^n \chi_{[t_i, t_{i+1})}(t) x_i,$$

where $n \in \mathbb{N}$, $0 = t_1 < t_2 < \dots < t_n < t_{n+1} = T$ and $x_i \in L^r_{\mathcal{F}_{t_i}}(\Omega; H)$, such that

$$|f - f_n|_{L^s_{\mathbb{F}}(0, T; L^r(\Omega; H))} < \frac{\varepsilon}{2}.$$

On the other hand, for each $\chi_{[t_i, t_{i+1})}$, we can find a $g_i \in C_0^\infty(t_i, t_{i+1})$ such that

$$|\chi_{[t_i, t_{i+1})} - g_i|_{L^s(0, T)} \leq \frac{\varepsilon}{2n(1 + |x_i|_{L^r(\Omega; H)}}.$$

Write $g = \sum_{i=1}^n g_i(t)x_i$. Then, it is clear that $g \in C_{0,\mathbb{F}}^\infty((0, T); L^r(\Omega; H))$. Moreover,

$$\begin{aligned} |f - g|_{L_{\mathbb{F}}^s(0, T; L^r(\Omega; H))} &\leq |f - f_n|_{L_{\mathbb{F}}^s(0, T; L^r(\Omega; H))} + |f_n - g|_{L_{\mathbb{F}}^s(0, T; L^r(\Omega; H))} \\ &< \frac{\varepsilon}{2} + \sum_{i=1}^n |\chi_{[t_i, t_{i+1})} x_i - g_i x_i|_{L_{\mathbb{F}}^s(0, T; L^r(\Omega; H))} < \varepsilon. \end{aligned}$$

This completes the proof of Lemma 2.2. \square

Fix any t_1 and t_2 satisfying $0 \leq t_2 < t_1 \leq T$, we recall the following known Riesz-type Representation Theorem (See [17, Corollary 2.3 and Remark 2.4]).

Lemma 2.3 *Assume that Y is a reflexive Banach space. Then, for any $r, s \in [1, \infty)$, it holds that*

$$(L_{\mathbb{F}}^r(t_2, t_1; L^s(\Omega; Y)))^* = L_{\mathbb{F}}^{r'}(t_2, t_1; L^{s'}(\Omega; Y^*)),$$

where $s' = s/(s-1)$ if $s \neq 1$; $s' = \infty$ if $s = 1$; and $r' = r/(r-1)$ if $r \neq 1$; $r' = \infty$ if $r = 1$.

Several more lemmas are in order.

Lemma 2.4 *Let $q \geq 2$. For any $(v_1(\cdot), v_2(\cdot), \eta) \in L_{\mathbb{F}}^1(t, T; L^q(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^q(\Omega; H)) \times L_{\mathcal{F}_t}^q(\Omega; H)$, the mild solution $z(\cdot) \in C_{\mathbb{F}}([t, T]; L^q(\Omega; H))$ of the equation (1.11), given by*

$$z(\cdot) = S(\cdot - t)\eta + \int_t^\cdot S(\cdot - \sigma)v_1(\sigma)d\sigma + \int_t^\cdot S(\cdot - \sigma)v_2(\sigma)dw(\sigma), \quad (2.2)$$

satisfies

$$\begin{aligned} &|z(\cdot)|_{C_{\mathbb{F}}([t, T]; L^q(\Omega; H))} \\ &\leq C \left| (v_1(\cdot), v_2(\cdot), \eta) \right|_{L_{\mathbb{F}}^1(t, T; L^q(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^q(\Omega; H)) \times L_{\mathcal{F}_t}^q(\Omega; H)}, \quad \forall t \in [0, T]. \end{aligned} \quad (2.3)$$

Proof: By (2.2), it is easy to see that $z(\cdot) \in C_{\mathbb{F}}([t, T]; L^q(\Omega; H))$. Also, by Lemma 2.1 and Minkowski's inequality, we have that

$$\begin{aligned} \mathbb{E}|z(s)|_H^q &= \mathbb{E} \left| S(s-t)\eta + \int_t^s S(s-\sigma)v_1(\sigma)d\sigma + \int_t^s S(s-\sigma)v_2(\sigma)dw(\sigma) \right|_H^q \\ &\leq C \left\{ \mathbb{E} \left| S(s-t)\eta \right|_H^q + \mathbb{E} \left| \int_t^s S(s-\sigma)v_1(\sigma)d\sigma \right|_H^q + \mathbb{E} \left[\int_t^s \left| S(s-\sigma)v_2(\sigma) \right|_H^2 d\sigma \right]^{\frac{q}{2}} \right\} \\ &\leq C \left\{ \mathbb{E} |\eta|_H^q + \mathbb{E} \left[\int_t^s |v_1(\sigma)|_H d\sigma \right]^q + \mathbb{E} \left[\int_t^s |v_2(\sigma)|_H^2 d\sigma \right]^{\frac{q}{2}} \right\} \\ &\leq C \left[\mathbb{E} |\eta|_H^q + |v_1(\cdot)|_{L_{\mathbb{F}}^q(\Omega; L^1(t, T; H))} + |v_2(\cdot)|_{L_{\mathbb{F}}^q(\Omega; L^2(t, T; H))}^2 \right] \\ &\leq C \left[\mathbb{E} |\eta|_H^q + |v_1(\cdot)|_{L_{\mathbb{F}}^1(t, T; L^q(\Omega; H))} + |v_2(\cdot)|_{L_{\mathbb{F}}^2(t, T; L^q(\Omega; H))}^2 \right], \end{aligned}$$

which gives (2.3). \square

Lemma 2.5 *Assume that $p \in (1, \infty]$, $q = \begin{cases} \frac{p}{p-1} & \text{if } p \in (1, \infty), \\ 1 & \text{if } p = \infty, \end{cases}$ $f_1 \in L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))$ and $f_2 \in L_{\mathbb{F}}^q(0, T; L^2(\Omega; H))$. Then there exists a monotonic sequence $\{h_n\}_{n=1}^\infty$ of positive numbers such that $\lim_{n \rightarrow \infty} h_n = 0$, and*

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \mathbb{E} \langle f_1(t), f_2(\tau) \rangle_H d\tau = \mathbb{E} \langle f_1(t), f_2(t) \rangle_H, \quad \text{a.e. } t \in [0, T]. \quad (2.4)$$

Proof: Write

$$\tilde{f}_2 = \begin{cases} f_2, & t \in [0, T], \\ 0, & t \in (T, 2T]. \end{cases}$$

Obviously, $\tilde{f}_2 \in L_{\mathbb{F}}^q(0, 2T; L^2(\Omega; H))$ and

$$|\tilde{f}_2|_{L_{\mathbb{F}}^q(0, 2T; L^2(\Omega; H))} = |\tilde{f}_2|_{L_{\mathbb{F}}^q(0, T; L^2(\Omega; H))} = |f_2|_{L_{\mathbb{F}}^q(0, T; L^2(\Omega; H))}.$$

By Lemma 2.2, for any $\varepsilon > 0$, one can find an $f_2^0 \in C_{\mathbb{F}}([0, 2T]; L^2(\Omega; H))$ such that

$$|\tilde{f}_2 - f_2^0|_{L_{\mathbb{F}}^q(0, 2T; L^2(\Omega; H))} \leq \varepsilon. \quad (2.5)$$

By the uniform continuity of $f_2^0(\cdot)$ in $L^2(\Omega; H)$, one can find a $\delta = \delta(\varepsilon) > 0$ such that

$$|f_2^0(s_1) - f_2^0(s_2)|_{L_{\mathcal{F}_T}^2(\Omega; H)} \leq \varepsilon, \quad \forall s_1, s_2 \in [0, 2T] \text{ satisfying } |s_1 - s_2| \leq \delta. \quad (2.6)$$

By means of (2.6), for each $h \leq \delta$, we have

$$\begin{aligned} & \int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), f_2^0(\tau) \rangle_H d\tau - \mathbb{E} \langle f_1(t), f_2^0(t) \rangle_H \right| dt \\ &= \frac{1}{h} \int_0^T \left| \int_t^{t+h} \mathbb{E} \langle f_1(t), f_2^0(\tau) - f_2^0(t) \rangle_H d\tau \right| dt \\ &\leq \frac{1}{h} \int_0^T \int_t^{t+h} |f_1(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)} |f_2^0(\tau) - f_2^0(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)} d\tau dt \\ &\leq \frac{\varepsilon}{h} \int_0^T \int_t^{t+h} |f_1(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)} d\tau dt = \varepsilon \int_0^T |f_1(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)} dt \leq C\varepsilon |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))}. \end{aligned} \quad (2.7)$$

Owing to (2.5), we find that

$$\begin{aligned} & \int_0^T \left| \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle_H - \mathbb{E} \langle f_1(t), f_2^0(t) \rangle_H \right| dt \\ &\leq |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))} |\tilde{f}_2 - f_2^0|_{L_{\mathbb{F}}^q(0, 2T; L^2(\Omega; H))} \leq \varepsilon |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))}. \end{aligned} \quad (2.8)$$

Further, utilizing (2.5) again, we see that

$$\begin{aligned} & \int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle_H d\tau - \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), f_2^0(\tau) \rangle_H d\tau \right| dt \\ &= \frac{1}{h} \int_0^T \left| \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) - f_2^0(\tau) \rangle_H d\tau \right| dt \\ &\leq \frac{1}{h} \int_0^T \int_t^{t+h} |f_1(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)} |\tilde{f}_2(\tau) - f_2^0(\tau)|_{L_{\mathcal{F}_T}^2(\Omega; H)} d\tau dt \\ &\leq \frac{1}{h} \left[\int_0^T \int_t^{t+h} |f_1(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)}^p d\tau dt \right]^{1/p} \left[\int_0^T \int_t^{t+h} |\tilde{f}_2(\tau) - f_2^0(\tau)|_{L_{\mathcal{F}_T}^2(\Omega; H)}^q d\tau dt \right]^{1/q} \\ &= |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))} \left[\frac{1}{h} \int_0^T \int_0^h |\tilde{f}_2(t+\tau) - f_2^0(t+\tau)|_{L_{\mathcal{F}_T}^2(\Omega; H)}^q d\tau dt \right]^{1/q} \\ &= |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))} \left[\frac{1}{h} \int_0^h \int_{\tau}^{T+\tau} |\tilde{f}_2(t) - f_2^0(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)}^q dt d\tau \right]^{1/q} \\ &\leq |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))} \left[\frac{1}{h} \int_0^h \int_0^T |\tilde{f}_2(t) - f_2^0(t)|_{L_{\mathcal{F}_T}^2(\Omega; H)}^q dt d\tau \right]^{1/q} \leq \varepsilon |f_1|_{L_{\mathbb{F}}^p(0, T; L^2(\Omega; H))}. \end{aligned} \quad (2.9)$$

From (2.7), (2.8) and (2.9), we conclude that

$$\int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle_H d\tau - \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle_H \right| dt \leq C\varepsilon |f_1|_{L_{\mathbb{F}}^p(0,T;L^2(\Omega;H))}.$$

Therefore,

$$\lim_{h \rightarrow 0} \int_0^T \left| \frac{1}{h} \int_t^{t+h} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle_H d\tau - \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle_H \right| dt = 0.$$

This implies that there exists a monotonic sequence $\{h_n\}_{n=1}^\infty$ of positive numbers with $\lim_{n \rightarrow \infty} h_n = 0$, such that

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle_H d\tau = \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle_H, \quad \text{a.e. } t \in [0, T].$$

By this and the definition of $\tilde{f}_2(\cdot)$, we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \mathbb{E} \langle f_1(t), f_2(\tau) \rangle_H d\tau &= \lim_{n \rightarrow \infty} \frac{1}{h_n} \int_t^{t+h_n} \mathbb{E} \langle f_1(t), \tilde{f}_2(\tau) \rangle_H d\tau = \mathbb{E} \langle f_1(t), \tilde{f}_2(t) \rangle_H \\ &= \mathbb{E} \langle f_1(t), f_2(t) \rangle_H, \quad \text{a.e. } t \in [0, T]. \end{aligned}$$

This completes the proof of Lemma 2.5. \square

Lemma 2.6 *For each $t \in [0, T]$, the following three conclusions hold:*

i) *If $u_2 = v_2 = 0$ in the equation (1.14), then there exists an operator $U(\cdot, t) \in \mathcal{L}(L_{\mathcal{F}_t}^4(\Omega; H), C_{\mathbb{F}}([t, T]; L^4(\Omega; H)))$ such that the solution to (1.14) can be represented as $x_2(\cdot) = U(\cdot, t)\xi_2$. Further, for any $t \in [0, T]$, $\xi \in L_{\mathcal{F}_t}^4(\Omega; H)$ and $\varepsilon > 0$, there is a $\delta \in (0, T - t)$ such that for any $s \in [t, t + \delta]$, it holds that*

$$|U(\cdot, t)\xi - U(\cdot, s)\xi|_{L_{\mathbb{F}}^\infty(s, T; L^4(\Omega; H))} < \varepsilon. \quad (2.10)$$

ii) *If $\xi_2 = 0$ and $v_2 = 0$ in the equation (1.14), then there exists an operator $V(\cdot, t) \in \mathcal{L}(L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)), C_{\mathbb{F}}([t, T]; L^4(\Omega; H)))$ such that the solution to (1.14) can be represented as $x_2(\cdot) = V(\cdot, t)u_2$.*

iii) *If $\xi_2 = 0$ and $u_2 = 0$ in the equation (1.14), then there exists an operator $W(\cdot, t) \in \mathcal{L}(L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)), C_{\mathbb{F}}([t, T]; L^4(\Omega; H)))$ such that the solution to (1.14) can be represented as $x_2(\cdot) = W(\cdot, t)v_2$.*

Proof: We prove only the first conclusion. Define $U(\cdot, t)$ as follows:

$$\begin{cases} U(\cdot, t) : L_{\mathcal{F}_t}^4(\Omega; H) \rightarrow C_{\mathbb{F}}([t, T]; L^4(\Omega; H)), \\ U(s, t)\xi_2 = x_2(s), \quad \forall s \in [t, T], \end{cases}$$

where $x_2(\cdot)$ is the mild solution to (1.14) with $u_2 = v_2 = 0$.

By Lemma 2.1 and Hölder's inequality, and noting $J, K \in L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, we obtain that

$$\begin{aligned} \mathbb{E}|x_2(s)|_H^4 &= \mathbb{E} \left| S(s-t)\xi_2 + \int_t^s S(s-\sigma)J(\sigma)x_2(\sigma)d\sigma + \int_t^s S(s-\sigma)K(\sigma)x_2(\sigma)dw(\sigma) \right|_H^4 \\ &\leq C \left\{ \mathbb{E} \left| S(s-t)\xi_2 \right|_H^4 + \mathbb{E} \left| \int_t^s S(s-\sigma)J(\sigma)x_2(\sigma)d\sigma \right|_H^4 \right. \\ &\quad \left. + \mathbb{E} \left[\int_t^s \left| S(s-\sigma)K(\sigma)x_2(\sigma) \right|_H^2 d\sigma \right]^2 \right\} \\ &\leq C \left[\mathbb{E}|\xi_2|_H^4 + \int_t^s \left(|J(\sigma)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\sigma)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 \right) \mathbb{E}|x_2(\sigma)|_H^4 d\sigma \right], \quad \forall s \in [t, T]. \end{aligned}$$

This, together with Gronwall's inequality, implies that

$$|x_2(s)|_{C_{\mathbb{F}}([t,T];L^4(\Omega;H))} \leq C|\xi_2|_{L^4_{\mathcal{F}_t}(\Omega;H)}.$$

Hence, $U(\cdot, t)$ is a bounded linear operator from $L^4_{\mathcal{F}_t}(\Omega; H)$ to $C_{\mathbb{F}}([t, T]; L^4(\Omega; H))$ and $U(\cdot, t)\xi_2$ solves the equation (1.14) with $u_2 = v_2 = 0$.

On the other hand, from the definition of $U(\cdot, t)$ and $U(\cdot, s)$, for each $r \in [s, T]$, we see that

$$U(r, t)\xi = S(r - t)\xi + \int_t^r S(r - \tau)J(\tau)U(\tau, t)\xi d\tau + \int_t^r S(r - \tau)K(\tau)U(\tau, t)\xi dw,$$

and

$$U(r, s)\xi = S(r - s)\xi + \int_s^r S(r - \tau)J(\tau)U(\tau, s)\xi d\tau + \int_s^r S(r - \tau)K(\tau)U(\tau, s)\xi dw.$$

Hence,

$$\begin{aligned} & \mathbb{E}|U(r, s)\xi - U(r, t)\xi|_H^4 \\ & \leq C\mathbb{E}\left|S(r - s)\xi - S(r - t)\xi\right|_H^4 + C\mathbb{E}\left|\int_s^r S(r - \tau)J(\tau)[U(\tau, s)\xi - U(\tau, t)\xi]ds\right|_H^4 + \\ & \quad + C\mathbb{E}\left|\int_s^r S(r - \tau)K(\tau)[U(\tau, s)\xi - U(\tau, t)\xi]dw\right|_H^4 + C\mathbb{E}\left|\int_t^s S(r - \tau)J(\tau)U(\tau, t)\xi d\tau\right|_H^4 \\ & \quad + C\mathbb{E}\left|\int_t^s S(r - \tau)K(\tau)U(\tau, t)\xi dw\right|_H^4 \\ & \leq C\mathbb{E}\left|S(r - s)\xi - S(r - t)\xi\right|_H^4 \\ & \quad + C\int_s^r \left(|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4\right)\mathbb{E}|U(\tau, s)\xi - U(\tau, t)\xi|_H^4 d\tau \\ & \quad + C\int_t^s \left(|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4\right)\mathbb{E}|U(\tau, t)\xi|_H^4 d\tau \\ & \leq C\int_s^r \left(|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4\right)\mathbb{E}|U(\tau, s)\xi - U(\tau, t)\xi|_H^4 d\tau \\ & \quad + C\mathbb{E}\left|S(r - s)\xi - S(r - t)\xi\right|_H^4 + C\int_t^s \left(|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4\right)d\tau\mathbb{E}|\xi|_H^4. \end{aligned}$$

Then, by Gronwall's inequality, we find that

$$\mathbb{E}|U(r, s)\xi - U(r, t)\xi|_H^4 \leq C\left[h(r, s, t) + \int_s^r h(\sigma, s, t)d\sigma\right],$$

where

$$h(r, s, t) = \mathbb{E}\left|S(r - s)\xi - S(r - t)\xi\right|_H^4 + \int_t^s \left(|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4\right)d\tau\mathbb{E}|\xi|_H^4.$$

Further, it is easy to see that

$$\left|\xi - S(s - t)\xi\right|_H^4 \leq C|\xi|_H^4.$$

By Lebesgue's dominated convergence theorem, we have

$$\lim_{s \rightarrow t+0} \mathbb{E}\left|\xi - S(s - t)\xi\right|_H^4 = 0.$$

Hence, there is a $\delta \in (0, T - t)$ such that (2.10) holds for any $s \in [t, t + \delta]$. This completes the proof of Lemma 2.6. \square

For any $t \in [0, T]$ and $\lambda \in \rho(A)$, consider the following two forward stochastic differential equations:

$$\begin{cases} dx_1^\lambda = (A_\lambda + J)x_1^\lambda ds + u_1 ds + Kx_1^\lambda dw(s) + v_1 dw(s) & \text{in } (t, T], \\ x_1^\lambda(t) = \xi_1 \end{cases} \quad (2.11)$$

and

$$\begin{cases} dx_2^\lambda = (A_\lambda + J)x_2^\lambda ds + u_2 ds + Kx_2^\lambda dw(s) + v_2 dw(s) & \text{in } (t, T], \\ x_2^\lambda(t) = \xi_2. \end{cases} \quad (2.12)$$

Here (ξ_1, u_1, v_1) (*resp.* (ξ_2, u_2, v_2)) is the same as that in (1.13) (*resp.* (1.14)). We have the following result:

Lemma 2.7 *The solutions of (2.11) and (2.12) satisfy*

$$\begin{cases} \lim_{\lambda \rightarrow \infty} x_1^\lambda(\cdot) = x_1(\cdot) \text{ in } C_{\mathbb{F}}([t, T]; L^4(\Omega; H)), \\ \lim_{\lambda \rightarrow \infty} x_2^\lambda(\cdot) = x_2(\cdot) \text{ in } C_{\mathbb{F}}([t, T]; L^4(\Omega; H)). \end{cases} \quad (2.13)$$

Here $x_1(\cdot)$ and $x_2(\cdot)$ are solutions of (1.13) and (1.14), respectively.

Proof: Clearly, for any $s \in [t, T]$, it holds that

$$\begin{aligned} & \mathbb{E}|x_1(s) - x_1^\lambda(s)|_H^4 \\ &= \mathbb{E} \left| \left[S(s-t) - S_\lambda(s-t) \right] \xi_1 + \int_t^s \left[S(s-\sigma)J(\sigma)x_1(\sigma) - S_\lambda(s-\sigma)J(\sigma)x_1^\lambda(\sigma) \right] d\sigma \right. \\ & \quad + \int_t^s \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] u_1(\sigma) d\sigma + \int_t^s \left[S(s-\sigma)K(\sigma)x_1(\sigma) - S_\lambda(s-\sigma)K(\sigma)x_1^\lambda(\sigma) \right] dw(\sigma) \\ & \quad \left. + \int_t^s \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] v_1(\sigma) dw(\sigma) \right|_H^4. \end{aligned}$$

Since A_λ is the Yosida approximation of A , one can find a positive constant $C = C(A, T)$, independent of λ , such that

$$|S_\lambda(\cdot)|_{L^\infty(0, T; \mathcal{L}(H))} \leq C. \quad (2.14)$$

Hence,

$$\begin{aligned} & \mathbb{E} \left| \int_t^s \left[S(s-\sigma)J(\sigma)x_1(\sigma) - S_\lambda(s-\sigma)J(\sigma)x_1^\lambda(\sigma) \right] d\sigma \right|_H^4 \\ & \leq C \mathbb{E} \int_t^s \left| \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] J(\sigma)x_1(\sigma) \right|_H^4 d\sigma + C \mathbb{E} \int_t^s \left| S_\lambda(s-\sigma)J(\sigma)[x_1(\sigma) - x_1^\lambda(\sigma)] \right|_H^4 d\sigma \\ & \leq C \mathbb{E} \left| \int_t^s \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] J(\sigma)x_1(\sigma) d\sigma \right|_H^4 + C \mathbb{E} \int_t^s |J(\sigma)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 |x_1(\sigma) - x_1^\lambda(\sigma)|_H^4 d\sigma. \end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned} & \mathbb{E} \left| \int_t^s \left[S(s-\sigma)K(\sigma)x_1(\sigma) - S_\lambda(s-\sigma)K(\sigma)x_1^\lambda(\sigma) \right] dw(\sigma) \right|_H^4 \\ & \leq C \mathbb{E} \int_t^s \left| \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] K(\sigma)x_1(\sigma) \right|_H^4 d\sigma + C \mathbb{E} \int_t^s \left| S_\lambda(s-\sigma)K(\sigma)[x_1(\sigma) - x_1^\lambda(\sigma)] \right|_H^4 d\sigma \\ & \leq C \mathbb{E} \int_t^s \left| \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] K(\sigma)x_1(\sigma) \right|_H^4 d\sigma + C \mathbb{E} \int_t^s |K(\sigma)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 |x_1(\sigma) - x_1^\lambda(\sigma)|_H^4 d\sigma. \end{aligned}$$

Hence, for $t \leq s \leq T$,

$$\mathbb{E} |x_1(s) - x_1^\lambda(s)|_H^4 \leq \Lambda(\lambda, s) + C \mathbb{E} \int_t^s \left(|J(\sigma)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\sigma)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 \right) |x_1(\sigma) - x_1^\lambda(\sigma)|_H^4 d\sigma.$$

Here

$$\begin{aligned} \Lambda(\lambda, s) = & C \mathbb{E} \left| \left[S(s-t) - S_\lambda(s-t) \right] \xi_1 \right|_H^4 + C \mathbb{E} \left| \int_t^s \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] u_1(\sigma) d\sigma \right|_H^4 \\ & + C \mathbb{E} \int_t^s \left| \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] v_1(\sigma) \right|_H^4 d\sigma \\ & + C \mathbb{E} \left| \int_t^s \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] J(\sigma) x_1(\sigma) d\sigma \right|_H^4 \\ & + C \mathbb{E} \int_t^s \left| \left[S(s-\sigma) - S_\lambda(s-\sigma) \right] K(\sigma) x_1(\sigma) \right|_H^4 d\sigma. \end{aligned}$$

By Gronwall's inequality, it follows that

$$\mathbb{E} |x_1(s) - x_1^\lambda(s)|_H^4 \leq \Lambda(\lambda, s) + C \int_t^s e^{C(s-\tau)} \Lambda(\lambda, \tau) d\tau, \quad t \leq s \leq T.$$

Since A_λ is the Yosida approximation of A , we see that $\lim_{\lambda \rightarrow \infty} \Lambda(\lambda, s) = 0$, which implies that

$$\lim_{\lambda \rightarrow \infty} |x_1^\lambda(\cdot) - x_1(\cdot)|_{C_F([t, T]; L^4(\Omega; H))} = 0.$$

This leads to the first equality in (2.13). The second equality in (2.13) can be proved similarly. This completes the proof of Lemma 2.7. \square

Lemma 2.8 *Let H be a separable Hilbert space. Then, for any $r \geq 1$, $\xi \in L_{\mathcal{F}_T}^r(\Omega; H)$ and $t \in [0, T)$, it holds that*

$$\lim_{s \rightarrow t^+} |\mathbb{E}(\xi | \mathcal{F}_s) - \mathbb{E}(\xi | \mathcal{F}_t)|_{L_{\mathcal{F}_T}^r(\Omega; H)} = 0. \quad (2.15)$$

Proof: Assume that $\xi = \sum_{i=1}^{\infty} \xi_i e_i$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of H . It is easy to see that

$$\mathbb{E} \left(\sum_{i=1}^{\infty} |\xi_i|^2 \right)^{r/2} < \infty.$$

Hence, for any $\varepsilon > 0$, there exists a $N > 0$ such that $\mathbb{E} \left(\sum_{i=N+1}^{\infty} |\xi_i|^2 \right)^{r/2} < \frac{\varepsilon^r}{3^r}$. Obviously, $\mathbb{E}(\xi | \mathcal{F}_t) =$

$\sum_{i=1}^{\infty} \mathbb{E}(\xi_i | \mathcal{F}_t) e_i$ for any $t \in [0, T)$. By

$$\begin{aligned} & \left(\sum_{i=N+1}^{\infty} |\mathbb{E}(\xi_i | \mathcal{F}_t)|^2 \right)^{r/2} = \left| \mathbb{E} \left(\sum_{i=N+1}^{\infty} \xi_i e_i \mid \mathcal{F}_t \right) \right|_H^r \\ & \leq \mathbb{E} \left(\left| \sum_{i=N+1}^{\infty} \xi_i e_i \right|_H^r \mid \mathcal{F}_t \right) = \mathbb{E} \left(\left(\sum_{i=N+1}^{\infty} |\xi_i|^2 \right)^{r/2} \mid \mathcal{F}_t \right), \quad \text{a.s.,} \end{aligned}$$

we see that

$$\mathbb{E}\left(\sum_{i=N+1}^{\infty} |\mathbb{E}(\xi_i | \mathcal{F}_t)|^2\right)^{r/2} \leq \mathbb{E}\left(\sum_{i=N+1}^{\infty} |\xi_i|^2\right)^{r/2} < \frac{\varepsilon^r}{3^r}, \quad \forall t \in [0, T].$$

On the other hand, since $\{\mathbb{E}(\xi_i | \mathcal{F}_t)\}_{t \in [0, T]}$ is an H -valued $\{\mathcal{F}_t\}_{t \in [0, T]}$ -martingale for each $i \in \mathbb{N}$, we conclude that there is an H -valued càdlàg process $\{x_i(t)\}_{t \in [0, T]}$ such that $x_i(t) = \mathbb{E}(\xi_i | \mathcal{F}_t)$, P -a.s. Now, for each $i \in \{1, 2, \dots, N\}$, by the fact that the family $\{\mathbb{E}(\xi_i | \mathcal{F}_t)\}_{t \in [0, T]}$ is uniformly r th power integrable, we can find a $\delta > 0$ such that for any $t \leq s \leq t + \delta$, it holds that $\mathbb{E}|x_i(t) - x_i(s)|^r < \frac{\varepsilon^r}{3^r N^r}$. Therefore, for any $t \leq s \leq t + \delta$, it holds that

$$\begin{aligned} & [\mathbb{E}|\mathbb{E}(\xi | \mathcal{F}_s) - \mathbb{E}(\xi | \mathcal{F}_t)|_H^r]^{1/r} \\ & \leq \left[\mathbb{E}\left(\sum_{i=N+1}^{\infty} |\mathbb{E}(\xi_i | \mathcal{F}_s)|^2\right)^{r/2}\right]^{1/r} + \left[\mathbb{E}\left(\sum_{i=N+1}^{\infty} |\mathbb{E}(\xi_i | \mathcal{F}_t)|^2\right)^{r/2}\right]^{1/r} + \sum_{i=1}^N \left(\mathbb{E}|x_i(t) - x_i(s)|^r\right)^{1/r} \\ & < \varepsilon, \end{aligned}$$

which completes the proof. \square

Lemma 2.9 *Assume that H_1 is a Hilbert space, and U is a nonempty subset of H_1 . If $F(\cdot) \in L^2_{\mathbb{F}}(0, T; H_1)$ and $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ such that*

$$\operatorname{Re} \mathbb{E} \int_0^T \langle F(t, \cdot), u(t, \cdot) - \bar{u}(t, \cdot) \rangle_{H_1} dt \leq 0, \quad (2.16)$$

holds for any $u(\cdot) \in \mathcal{U}[0, T]$ satisfying $u(\cdot) - \bar{u}(\cdot) \in L^2_{\mathbb{F}}(0, T; L^2(\Omega; H_1))$, then, for any point $u \in U$, the following pointwise inequality holds:

$$\operatorname{Re} \langle F(t, \omega), u - \bar{u}(t, \omega) \rangle_{H_1} \leq 0, \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega. \quad (2.17)$$

Proof: We use the contradiction argument. Suppose that the inequality (2.17) did not hold. Then, there would exist a $u_0 \in U$ and an $\varepsilon > 0$ such that

$$\alpha_{\varepsilon} \triangleq \int_{\Omega} \int_0^T \chi_{\Lambda_{\varepsilon}}(t, \omega) dt dP > 0,$$

where $\Lambda_{\varepsilon} \triangleq \{(t, \omega) \in [0, T] \times \Omega \mid \operatorname{Re} \langle F(t, \omega), u_0 - \bar{u}(t, \omega) \rangle_{H_1} \geq \varepsilon\}$, and $\chi_{\Lambda_{\varepsilon}}$ is the characteristic function of Λ_{ε} . For any $m \in \mathbb{N}$, define $\Lambda_{\varepsilon, m} \triangleq \Lambda_{\varepsilon} \cap \{(t, \omega) \in [0, T] \times \Omega \mid |\bar{u}(t, \omega)|_{H_1} \leq m\}$. It is clear that $\lim_{m \rightarrow \infty} \Lambda_{\varepsilon, m} = \Lambda_{\varepsilon}$. Hence, there is an $m_{\varepsilon} \in \mathbb{N}$ such that

$$\int_{\Omega} \int_0^T \chi_{\Lambda_{\varepsilon, m}}(t, \omega) dt dP > \frac{\alpha_{\varepsilon}}{2} > 0, \quad \forall m \geq m_{\varepsilon}.$$

Since $\langle F(\cdot), u_0 - \bar{u}(\cdot) \rangle_{H_1}$ is $\{\mathcal{F}_t\}$ -adapted, so is the process $\chi_{\Lambda_{\varepsilon, m}}(\cdot)$. Define

$$\hat{u}_{\varepsilon, m}(t, \omega) = u_0 \chi_{\Lambda_{\varepsilon, m}}(t, \omega) + \bar{u}(t, \omega) \chi_{\Lambda_{\varepsilon, m}^c}(t, \omega), \quad (t, \omega) \in [0, T] \times \Omega.$$

Noting that $|\bar{u}(\cdot)|_{H_1} \leq m$ on $\Lambda_{\varepsilon,m}$, we see that $\hat{u}_{\varepsilon,m}(\cdot) \in \mathcal{U}[0, T]$ and satisfies $\hat{u}_{\varepsilon,m}(\cdot) - \bar{u}(\cdot) \in L^2_{\mathbb{F}}(0, T; H)$. Hence, for any $m \geq m_{\varepsilon}$, we obtain that

$$\begin{aligned} \operatorname{Re} \mathbb{E} \int_0^T \langle F(t), \hat{u}_{\varepsilon,m}(t) - \bar{u}(t) \rangle_{H_1} dt &= \int_{\Omega} \int_0^T \chi_{\Lambda_{\varepsilon,m}}(t, \omega) \operatorname{Re} \langle F(t, \omega), u_0 - \bar{u}(t, \omega) \rangle_{H_1} dt dP \\ &\geq \varepsilon \int_{\Omega} \int_0^T \chi_{\Lambda_{\varepsilon,m}}(t, \omega) dt dP \\ &\geq \frac{\varepsilon \alpha_{\varepsilon}}{2} > 0, \end{aligned}$$

which contradicts (2.16). This completes the proof of Lemma 2.9. \square

3 Well-posedness of the vector-valued BSEs

This section is devoted to proving the following result.

Theorem 3.1 *For any $p \in (1, 2]$, $y_T \in L^p_{\mathcal{F}_T}(\Omega; H)$, $f(\cdot, \cdot, \cdot) : [0, T] \times H \times H \rightarrow H$ satisfying (1.9), the equation (1.8) admits one and only one transposition solution $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0, T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}(0, T; L^p(\Omega; H))$. Furthermore,*

$$\begin{aligned} &|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([0, T]; L^p(\Omega; H)) \times L^2_{\mathbb{F}}(0, T; L^p(\Omega; H))} \\ &\leq C \left[|f(\cdot, 0, 0)|_{L^1_{\mathbb{F}}(t, T; L^p(\Omega; H))} + |y_T|_{L^p_{\mathcal{F}_T}(\Omega; H)} \right], \quad \forall t \in [0, T]. \end{aligned} \quad (3.1)$$

Proof: We borrow some ideas from the proof of [18, Theorem 3.1]. The proof is divided into five steps. In the first four steps, we study (1.8) for a special case, in which $f(\cdot, \cdot, \cdot)$ is independent of y and Y . More precisely, for any $y_T \in L^p_{\mathcal{F}_T}(\Omega; H)$ and $f(\cdot) \in L^1_{\mathbb{F}}(0, T; L^p(\Omega; H))$, we consider first the following equation:

$$\begin{cases} dy(t) = -A^*y(t)dt + f(t)dt + Y(t)dw(t) & \text{in } [0, T], \\ y(T) = y_T. \end{cases} \quad (3.2)$$

In the last step, we deal with (1.8) for the general case by the fixed point technique.

Step 1. For any $t \in [0, T]$, we define a linear functional ℓ (depending on t) on the Banach space $L^1_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^2_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^q_{\mathcal{F}_t}(\Omega; H)$ as follows (Recall that $q = \frac{p}{p-1}$):

$$\begin{aligned} \ell(v_1(\cdot), v_2(\cdot), \eta) &= \mathbb{E} \langle z(T), y_T \rangle_H - \mathbb{E} \int_t^T \langle z(s), f(s) \rangle_H ds, \\ \forall (v_1(\cdot), v_2(\cdot), \eta) &\in L^1_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^2_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^q_{\mathcal{F}_t}(\Omega; H), \end{aligned} \quad (3.3)$$

where $z(\cdot) \in C_{\mathbb{F}}([t, T]; L^q(\Omega; H))$ solves the equation (1.11).

By means of the Hölder inequality and Lemma 2.4, it is easy to show that

$$\begin{aligned} &|\ell(v_1(\cdot), v_2(\cdot), \eta)| \\ &\leq |z(T)|_{L^q_{\mathcal{F}_T}(\Omega; H)} |y_T|_{L^p_{\mathcal{F}_T}(\Omega; H)} + |z(\cdot)|_{C_{\mathbb{F}}([t, T]; L^q(\Omega; H))} |f|_{L^1_{\mathbb{F}}(t, T; L^p(\Omega; H))} \\ &\leq C \left[|f(\cdot)|_{L^1_{\mathbb{F}}(t, T; L^p(\Omega; H))} + |y_T|_{L^p_{\mathcal{F}_T}(\Omega; H)} \right] \\ &\quad \times |(v_1(\cdot), v_2(\cdot), \eta)|_{L^1_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^2_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^q_{\mathcal{F}_t}(\Omega; H)}, \quad \forall t \in [0, T], \end{aligned} \quad (3.4)$$

where the positive constant $C = C(T, A)$ is independent of t . From (3.4), it follows that ℓ is a bounded linear functional on $L_{\mathbb{F}}^1(t, T; L^q(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^q(\Omega; H)) \times L_{\mathcal{F}_t}^q(\Omega; H)$. By Lemma 2.3, there exist $y^t(\cdot) \in L_{\mathbb{F}}^\infty(t, T; L^p(\Omega; H))$, $Y^t(\cdot) \in L_{\mathbb{F}}^2(t, T; L^p(\Omega; H))$ and $\xi^t \in L_{\mathcal{F}_t}^p(\Omega; H)$ such that

$$\begin{aligned} & \mathbb{E} \langle z(T), y_T \rangle_H - \mathbb{E} \int_t^T \langle z(\tau), f(\tau) \rangle_H d\tau \\ &= \mathbb{E} \int_t^T \langle v_1(\tau), y^t(\tau) \rangle_H d\tau + \mathbb{E} \int_t^T \langle v_2(\tau), Y^t(\tau) \rangle_H d\tau + \mathbb{E} \langle \eta, \xi^t \rangle_H. \end{aligned} \quad (3.5)$$

It is clear that $\xi^T = y_T$. Furthermore, there is a positive constant $C = C(T, A)$, independent of t , such that

$$\begin{aligned} & |(y^t(\cdot), Y^t(\cdot), \xi^t)|_{L_{\mathbb{F}}^\infty(t, T; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^p(\Omega; H)) \times L_{\mathcal{F}_t}^p(\Omega; H)} \\ & \leq C \left[|f(\cdot)|_{L_{\mathbb{F}}^1(t, T; L^p(\Omega; H))} + |y_T|_{L_{\mathcal{F}_T}^p(\Omega; H)} \right], \quad \forall t \in [0, T]. \end{aligned} \quad (3.6)$$

Step 2. Note that the function $(y^t(\cdot), Y^t(\cdot))$ obtained in Step 1 may depend on t . In this step, we show the time consistency of $(y^t(\cdot), Y^t(\cdot))$, that is, for any t_1 and t_2 satisfying $0 \leq t_2 \leq t_1 \leq T$, it holds that

$$(y^{t_2}(\tau, \omega), Y^{t_2}(\tau, \omega)) = (y^{t_1}(\tau, \omega), Y^{t_1}(\tau, \omega)), \quad \text{a.e. } (\tau, \omega) \in [t_1, T] \times \Omega. \quad (3.7)$$

Since the solution $z(\cdot)$ of (1.11) depends on t , we also denote it by $z^t(\cdot)$ whenever there exists a possible confusion. To show (3.7), we fix arbitrarily $\varrho(\cdot) \in L_{\mathbb{F}}^1(t_1, T; L^q(\Omega; H))$ and $\varsigma(\cdot) \in L_{\mathbb{F}}^2(t_1, T; L^q(\Omega; H))$, and choose first $t = t_1$, $\eta = 0$, $v_1(\cdot) = \varrho(\cdot)$ and $v_2(\cdot) = \varsigma(\cdot)$ in (1.11). From (3.5), we obtain that

$$\begin{aligned} & \mathbb{E} \langle z^{t_1}(T), y_T \rangle_H - \mathbb{E} \int_{t_1}^T \langle z^{t_1}(\tau), f(\tau) \rangle_H d\tau \\ &= \mathbb{E} \int_{t_1}^T \langle \varrho(\tau), y^{t_1}(\tau) \rangle_H d\tau + \mathbb{E} \int_{t_1}^T \langle \varsigma(\tau), Y^{t_1}(\tau) \rangle_H d\tau. \end{aligned} \quad (3.8)$$

Then, choose $t = t_2$, $\eta = 0$, $v_1(t, \omega) = \chi_{[t_1, T]}(t) \varrho(t, \omega)$ and $v_2(t, \omega) = \chi_{[t_1, T]}(t) \varsigma(t, \omega)$ in (1.11). It is clear that

$$z^{t_2}(\cdot) = \begin{cases} z^{t_1}(\cdot), & t \in [t_1, T], \\ 0, & t \in [t_2, t_1). \end{cases}$$

From the equality (3.5), it follows that

$$\begin{aligned} & \mathbb{E} \langle z^{t_1}(T), y_T \rangle_H - \mathbb{E} \int_{t_1}^T \langle z^{t_1}(\tau), f(\tau) \rangle_H d\tau \\ &= \mathbb{E} \int_{t_1}^T \langle \varrho(\tau), y^{t_2}(\tau) \rangle_H d\tau + \mathbb{E} \int_{t_1}^T \langle \varsigma(\tau), Y^{t_2}(\tau) \rangle_H d\tau. \end{aligned} \quad (3.9)$$

Combining (3.8) and (3.9), we obtain that

$$\begin{aligned} & \mathbb{E} \int_{t_1}^T \langle \varrho(\tau), y^{t_1}(\tau) - y^{t_2}(\tau) \rangle_H d\tau + \mathbb{E} \int_{t_1}^T \langle \varsigma(\tau), Y^{t_1}(\tau) - Y^{t_2}(\tau) \rangle_H d\tau = 0, \\ & \quad \forall \varrho(\cdot) \in L_{\mathbb{F}}^1(t_1, T; L^q(\Omega; H)), \quad \varsigma(\cdot) \in L_{\mathbb{F}}^2(t_1, T; L^q(\Omega; H)). \end{aligned}$$

This yields the desired equality (3.7).

Put

$$y(t, \omega) = y^0(t, \omega), \quad Y(t, \omega) = Y^0(t, \omega), \quad \forall (t, \omega) \in [0, T] \times \Omega. \quad (3.10)$$

From (3.7), it follows that

$$(y^t(\tau, \omega), Y^t(\tau, \omega)) = (y(\tau, \omega), Y(\tau, \omega)), \quad \text{a.e. } (\tau, \omega) \in [t, T] \times \Omega. \quad (3.11)$$

Combining (3.5) and (3.11), we end up with

$$\begin{aligned} & \mathbb{E} \langle z(T), y_T \rangle_H - \mathbb{E} \langle \eta, \xi^t \rangle_H \\ &= \mathbb{E} \int_t^T \langle z(\tau), f(\tau) \rangle_H d\tau + \mathbb{E} \int_t^T \langle v_1(\tau), y(\tau) \rangle_H d\tau + \mathbb{E} \int_t^T \langle v_2(\tau), Y(\tau) \rangle_H d\tau, \\ & \quad \forall (v_1(\cdot), v_2(\cdot), \eta) \in L^1_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^2_{\mathbb{F}}(t, T; L^q(\Omega; H)) \times L^q_{\mathcal{F}_t}(\Omega; H). \end{aligned} \quad (3.12)$$

Step 3. We show in this step that ξ^t has a càdlàg modification.

First of all, we claim that, for each $t \in [0, T]$,

$$\mathbb{E} \left(S^*(T-t)y_T - \int_t^T S^*(s-t)f(s)ds \mid \mathcal{F}_t \right) = \xi^t, \quad \mathbb{P}\text{-a.s.} \quad (3.13)$$

To prove this, we note that for any $\eta \in L^q_{\mathcal{F}_t}(\Omega; H)$, $v_1 = 0$ and $v_2 = 0$, the corresponding solution to (1.11) is given by $z(s) = S(s-t)\eta$ for $s \in [t, T]$. Hence, by (3.12), we obtain that

$$\mathbb{E} \langle S(T-t)\eta, y_T \rangle_H - \mathbb{E} \langle \eta, \xi^t \rangle_H = \mathbb{E} \int_t^T \langle S(s-t)\eta, f(s) \rangle_H ds. \quad (3.14)$$

Noting that

$$\mathbb{E} \langle S(T-t)\eta, y_T \rangle_H = \mathbb{E} \langle \eta, S^*(T-t)y_T \rangle_H = \mathbb{E} \langle \eta, \mathbb{E}(S^*(T-t)y_T \mid \mathcal{F}_t) \rangle_H$$

and

$$\mathbb{E} \int_t^T \langle S(s-t)\eta, f(s) \rangle_H ds = \mathbb{E} \left\langle \eta, \int_t^T S^*(s-t)f(s)ds \right\rangle_H = \mathbb{E} \left\langle \eta, \mathbb{E} \left(\int_t^T S^*(s-t)f(s)ds \mid \mathcal{F}_t \right) \right\rangle_H,$$

by (3.14), we conclude that

$$\mathbb{E} \left\langle \eta, \mathbb{E} \left(S^*(T-t)y_T - \int_t^T S^*(s-t)f(s)ds \mid \mathcal{F}_t \right) - \xi^t \right\rangle_H = 0, \quad \forall \eta \in L^q_{\mathcal{F}_t}(\Omega; H). \quad (3.15)$$

Clearly, (3.13) follows from (3.15) immediately.

In the rest of this step, we show that the process

$$\left\{ \mathbb{E} \left(S^*(T-t)y_T - \int_t^T S^*(s-t)f(s)ds \mid \mathcal{F}_t \right) \right\}_{t \in [0, T]}$$

has a càdlàg modification. Unlike the case that H is a finite dimensional space, the proof of this fact (in the infinite dimensional space) is quite technical.

Noting that H is not assumed to be separable (in this section), we are going to construct a separable subspace of H as our working space. For this purpose, noting that the set of simple

functions is dense in $L^p_{\mathcal{F}_T}(\Omega; H)$, we conclude that there exists a sequence $\{y^m\}_{m=1}^\infty \subset L^2_{\mathcal{F}_T}(\Omega; H)$ satisfying the following two conditions:

$$1) \quad y^m = \sum_{k=1}^{N_m} \alpha_k^m \chi_{\Omega_k^m}(\omega), \text{ where } N_m \in \mathbb{N}, \alpha_k^m \in H \text{ and } \Omega_k^m \in \mathcal{F}_T \text{ with } \{\Omega_k^m\}_{k=1}^{N_m} \text{ to be a partition}$$

of Ω ; and

$$2) \quad \lim_{m \rightarrow \infty} |y^m - y_T|_{L^p_{\mathcal{F}_T}(\Omega; H)} = 0.$$

Likewise, since the set of simple adapted processes is dense in $L^p_{\mathbb{F}}(\Omega; L^1(0, T; H))$, there exists a sequence $\{f^m\}_{m=1}^\infty \subset L^1_{\mathbb{F}}(0, T; L^p(\Omega; H))$ satisfying the following two conditions:

$$i) \quad f^m = \sum_{j=1}^{L_m} \sum_{k=1}^{M_j^m} \alpha_{j,k}^m \chi_{\Omega_{j,k}^m}(\omega) \chi_{[t_j^m, t_{j+1}^m)}(t), \text{ where } L_m \in \mathbb{N}, M_j^m \in \mathbb{N}, \alpha_{j,k}^m \in H, \Omega_{j,k}^m \in \mathcal{F}_{t_j^m} \text{ with}$$

$\{\Omega_{j,k}^m\}_{k=1}^{M_j^m}$ being a partition of Ω , and $0 = t_1^m < t_2^m \cdots < t_{J_m}^m < t_{J_m+1}^m = T$; and

$$ii) \quad \lim_{m \rightarrow \infty} |f^m - f|_{L^1_{\mathbb{F}}(0, T; L^p(\Omega; H))} = 0.$$

Denote by Ξ the set of all the above elements α_k^m ($k = 1, 2, \dots, N_m$; $m = 1, 2, \dots$) and $\alpha_{j,k}^m$ ($k = 1, 2, \dots, M_j^m$; $j = 1, 2, \dots, L_m$; $m = 1, 2, \dots$) in H , and by \tilde{H} the closure of $\text{span } \Xi$ under the topology of H . Clearly, \tilde{H} is a separable closed subspace of H , and hence, \tilde{H} itself is also a Hilbert space.

Recall that for any $\lambda \in \rho(A)$, the bounded operator A_λ (resp. A_λ^*) generates a C_0 -group $\{S_\lambda(t)\}_{t \in \mathbb{R}}$ (resp. $\{S_\lambda^*(t)\}_{t \in \mathbb{R}}$) on H .

For each $m \in \mathbb{N}$ and $t \in [0, T]$, put

$$\xi_{\lambda, m}^t \triangleq \mathbb{E} \left(S_\lambda^*(T-t) y^m - \int_t^T S_\lambda^*(s-t) f^m(s) ds \mid \mathcal{F}_t \right) \quad (3.16)$$

and

$$X_\lambda^m(t) \triangleq S_\lambda^*(t) \xi_{\lambda, m}^t - \int_0^t S_\lambda^*(s) f^m(s) ds. \quad (3.17)$$

We claim that $\{X_\lambda^m(t)\}$ is an \tilde{H} -valued $\{\mathcal{F}_t\}$ -martingale. In fact, for any $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 \leq \tau_2$, by (3.16) and (3.17), it follows that

$$\begin{aligned} & \mathbb{E}(X_\lambda^m(\tau_2) \mid \mathcal{F}_{\tau_1}) \\ &= \mathbb{E} \left(S_\lambda^*(\tau_2) \xi_{\lambda, m}^{\tau_2} - \int_0^{\tau_2} S_\lambda^*(s) f^m(s) ds \mid \mathcal{F}_{\tau_1} \right) \\ &= \mathbb{E} \left[\mathbb{E} \left(S_\lambda^*(T) y^m - \int_{\tau_2}^T S_\lambda^*(s) f^m(s) ds \mid \mathcal{F}_{\tau_2} \right) - \int_0^{\tau_2} S_\lambda^*(s) f^m(s) ds \mid \mathcal{F}_{\tau_1} \right] \\ &= \mathbb{E} \left(S_\lambda^*(T) y^m - \int_0^T S_\lambda^*(s) f^m(s) ds \mid \mathcal{F}_{\tau_1} \right) \\ &= S_\lambda^*(\tau_1) \mathbb{E} \left(S_\lambda^*(T-\tau_1) y^m - \int_{\tau_1}^T S_\lambda^*(s-\tau_1) f^m(s) ds \mid \mathcal{F}_{\tau_1} \right) - \int_0^{\tau_1} S_\lambda^*(s) f^m(s) ds \\ &= S_\lambda^*(\tau_1) \xi_{\lambda, m}^{\tau_1} - \int_0^{\tau_1} S_\lambda^*(s) f^m(s) ds \\ &= X_\lambda^m(\tau_1), \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (3.18)$$

as desired.

Now, since $\{X_\lambda^m(t)\}_{0 \leq t \leq T}$ is an \tilde{H} -valued \mathbb{F} -martingale, it enjoys a càdlàg modification, and hence so does the following process

$$\{\xi_{\lambda,m}^t\}_{0 \leq t \leq T} = \left\{ S_\lambda^*(-t) \left[X_\lambda^m(t) + \int_0^t S_\lambda^*(s) f^m(s) ds \right] \right\}_{0 \leq t \leq T}.$$

Here we have used the fact that $\{S_\lambda^*(t)\}_{t \in \mathbb{R}}$ is a C_0 -group on H . We still use $\{\xi_{\lambda,m}^t\}_{0 \leq t \leq T}$ to stand for its càdlàg modification.

From (3.13) and (3.16), it follows that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \|\xi_\cdot - \xi_{\lambda,m}\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} \\ &= \lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \left\| \mathbb{E} \left(S^*(T-\cdot) y_T - \int_\cdot^T S^*(s-\cdot) f(s) ds \mid \mathcal{F}_\cdot \right) \right. \\ & \quad \left. - \mathbb{E} \left(S_\lambda^*(T-\cdot) y^m - \int_\cdot^T S_\lambda^*(s-\cdot) f^m(s) ds \mid \mathcal{F}_\cdot \right) \right\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} \\ &\leq \lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \left\| S^*(T-\cdot) y_T - S_\lambda^*(T-\cdot) y^m \right\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} \\ & \quad + \lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \left\| \int_\cdot^T S^*(s-\cdot) f(s) ds - \int_\cdot^T S_\lambda^*(s-\cdot) f^m(s) ds \right\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))}. \end{aligned} \quad (3.19)$$

Let us prove the right hand side of (3.19) equals zero. First, we prove

$$\lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \left\| S^*(T-\cdot) y_T - S_\lambda^*(T-\cdot) y^m \right\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} = 0. \quad (3.20)$$

Since $\{S(t)\}_{t \geq 0}$ is a C_0 -semigroup, for any $\varepsilon > 0$, there is an $M > 0$ such that for any $m > M$, it holds that

$$\|S^*(T-\cdot) y_T - S^*(T-\cdot) y^m\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} < \frac{\varepsilon}{2}.$$

On the other hand, by the property of Yosida approximations, we deduce that for any $\alpha \in H$, it holds that $\lim_{\lambda \rightarrow \infty} \|S^*(T-\cdot) \alpha - S_\lambda^*(T-\cdot) \alpha\|_{L^\infty(0,T;H)} = 0$. Thus, there is a $\Lambda = \Lambda(m) > 0$ such that for any $\lambda > \Lambda$, it holds that

$$\|S^*(T-\cdot) \alpha_k^m - S_\lambda^*(T-\cdot) \alpha_k^m\|_{L^\infty(0,T;H)} < \frac{\varepsilon}{2N_m}, \quad k = 1, 2, \dots, N_m,$$

which implies that

$$\|S^*(T-\cdot) y^m - S_\lambda^*(T-\cdot) y^m\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} \leq \sum_{k=1}^{N_m} \|S^*(T-\cdot) \alpha_k^m - S_\lambda^*(T-\cdot) \alpha_k^m\|_{L^\infty(0,T;H)} < \frac{\varepsilon}{2}.$$

Therefore, for each $m > M$, there is a $\Lambda = \Lambda(m)$ such that when $\lambda > \Lambda(m)$, it holds that

$$\begin{aligned} & \left\| S^*(T-\cdot) y_T - S_\lambda^*(T-\cdot) y^m \right\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} \\ &\leq \|S^*(T-\cdot) y_T - S^*(T-\cdot) y^m\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} + \|S^*(T-\cdot) y^m - S_\lambda^*(T-\cdot) y^m\|_{L_{\mathbb{F}}^\infty(0,T;L^p(\Omega;H))} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This gives (3.20).

Further, we show that

$$\lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \left| \int_{\cdot}^T S^*(s - \cdot) f(s) ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) f^m(s) ds \right|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} = 0. \quad (3.21)$$

For any $\varepsilon > 0$, there is a $M^* > 0$ such that for any $m > M^*$,

$$\left| \int_{\cdot}^T S^*(s - \cdot) f(s) ds - \int_{\cdot}^T S^*(s - \cdot) f^m(s) ds \right|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} < \frac{\varepsilon}{2}.$$

By the property of Yosida approximations again, we know that for any $\alpha \in H$, it holds that

$$\lim_{\lambda \rightarrow \infty} \left| \int_{\cdot}^T S^*(s - \cdot) \alpha ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) \alpha ds \right|_{L^{\infty}(0, T; H)} = 0.$$

Thus, there is a $\Lambda^* = \Lambda^*(m) > 0$ such that for any $\lambda > \Lambda^*$,

$$\left| \int_{\cdot}^T S^*(s - \cdot) \alpha_{j,k}^m ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) \alpha_{j,k}^m ds \right|_{L^{\infty}(0, T; H)} < \frac{\varepsilon}{2J_m \max(M_1^m, M_2^m, \dots, M_{J_m}^m)},$$

$j = 1, 2, \dots, L_m; k = 1, 2, \dots, M_j^m.$

This implies that

$$\begin{aligned} & \left| \int_{\cdot}^T S^*(s - \cdot) f^m(s) ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) f^m(s) ds \right|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} \\ & \leq \sum_{j=1}^{L_m} \sum_{k=1}^{M_j^m} \left| \int_{\cdot}^T S^*(s - \cdot) \alpha_{j,k}^m ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) \alpha_{j,k}^m ds \right|_{L^{\infty}(0, T; H)} < \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, for any $m > M^*$ and $\lambda > \Lambda^* = \Lambda^*(m)$, we have

$$\begin{aligned} & \left| \int_{\cdot}^T S^*(s - \cdot) f(s) ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) f^m(s) ds \right|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} \\ & \leq \left| \int_{\cdot}^T S^*(s - \cdot) f(s) ds - \int_{\cdot}^T S^*(s - \cdot) f^m(s) ds \right|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} \\ & \quad + \left| \int_{\cdot}^T S^*(s - \cdot) f^m(s) ds - \int_{\cdot}^T S_{\lambda}^*(s - \cdot) f^m(s) ds \right|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} \\ & < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This gives (3.21).

By (3.19), (3.20) and (3.21), we obtain that $\lim_{m \rightarrow \infty} \lim_{\lambda \rightarrow \infty} |\xi_{\cdot} - \xi_{\lambda, m}|_{L_{\mathbb{F}}^{\infty}(0, T; L^p(\Omega; H))} = 0$. Recalling that $\xi_{\lambda, m} \in D_{\mathbb{F}}([0, T]; L^p(\Omega; H))$, we deduce that ξ_{\cdot} enjoys a càdlàg modification.

Step 4. In this step, we show that, for a.e. $t \in [0, T]$,

$$\xi^t = y(t), \quad \mathbb{P}\text{-a.s.} \quad (3.22)$$

We consider first the case that $p = 2$ and fix any $\gamma \in L_{\mathcal{F}_{t_2}}^2(\Omega; H)$. Choosing $t = t_2$, $v_1(\cdot) = 0$, $v_2(\cdot) = 0$ and $\eta = (t_1 - t_2)\gamma$ in (1.11), utilizing (3.12), we obtain that

$$\mathbb{E} \langle S(T - t_2)(t_1 - t_2)\gamma, y_T \rangle_H - \mathbb{E} \langle (t_1 - t_2)\gamma, \xi^{t_2} \rangle_H = \mathbb{E} \int_{t_2}^T \langle S(\tau - t_2)(t_1 - t_2)\gamma, f(\tau) \rangle_H d\tau. \quad (3.23)$$

Choosing $t = t_2$, $v_1(\tau, \omega) = \chi_{[t_2, t_1]}(\tau)\gamma(\omega)$, $v_2(\cdot) = 0$ and $\eta = 0$ in (1.11), utilizing (3.12) again, we find that

$$\begin{aligned} & \mathbb{E} \left\langle \int_{t_2}^T S(T-s) \chi_{[t_2, t_1]}(s) \gamma ds, y_T \right\rangle_H \\ &= \mathbb{E} \int_{t_2}^{t_1} \left\langle \int_{t_2}^{\tau} S(\tau-s) \gamma ds, f(\tau) \right\rangle_H d\tau + \mathbb{E} \int_{t_1}^T \left\langle S(\tau-t_1) \int_{t_2}^{t_1} S(t_1-s) \gamma ds, f(\tau) \right\rangle_H d\tau \\ & \quad + \mathbb{E} \int_{t_2}^{t_1} \langle \gamma, y(\tau) \rangle_H d\tau. \end{aligned} \quad (3.24)$$

From (3.23) and (3.24), we find

$$\begin{aligned} & \mathbb{E} \langle \gamma, \xi^{t_2} \rangle_H \\ &= \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \gamma, y(\tau) \rangle_H d\tau + \mathbb{E} \langle S(T-t_2) \gamma, y_T \rangle_H - \frac{1}{t_1 - t_2} \mathbb{E} \left\langle \int_{t_2}^T S(T-\tau) \chi_{[t_2, t_1]}(\tau) \gamma d\tau, y_T \right\rangle_H \\ & \quad - \mathbb{E} \int_{t_2}^T \langle S(\tau-t_2) \gamma, f(\tau) \rangle_H d\tau + \frac{1}{t_1 - t_2} \mathbb{E} \int_{t_2}^{t_1} \left\langle \int_{t_2}^{\tau} S(\tau-s) \gamma, f(\tau) \right\rangle_H d\tau \\ & \quad + \frac{1}{t_1 - t_2} \mathbb{E} \int_{t_1}^T \left\langle S(\tau-t_1) \int_{t_2}^{t_1} S(t_1-s) \gamma ds, f(\tau) \right\rangle_H d\tau. \end{aligned} \quad (3.25)$$

Now we analyze the terms in the right hand side of (3.25) one by one. First, it is easy to show that

$$\lim_{t_1 \rightarrow t_2+0} \frac{1}{t_1 - t_2} \mathbb{E} \int_{t_2}^{t_1} \left\langle \int_{t_2}^{\tau} S(s-t_2) \gamma, f(\tau) \right\rangle_H d\tau = 0, \quad \forall \gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega; H). \quad (3.26)$$

Further,

$$\begin{aligned} & \lim_{t_1 \rightarrow t_2+0} \frac{1}{t_1 - t_2} \mathbb{E} \left\langle \int_{t_2}^T S(T-\tau) \chi_{[t_2, t_1]}(\tau) \gamma d\tau, y_T \right\rangle_H \\ &= \lim_{t_1 \rightarrow t_2+0} \frac{1}{t_1 - t_2} \mathbb{E} \left\langle \int_{t_2}^{t_1} S(T-\tau) \gamma d\tau, y_T \right\rangle_H \\ &= \mathbb{E} \langle S(T-t_2) \gamma, y_T \rangle_H. \end{aligned} \quad (3.27)$$

Utilizing the semigroup property of $\{S(t)\}_{t \geq 0}$, we have

$$\lim_{t_1 \rightarrow t_2+0} \frac{1}{t_1 - t_2} \mathbb{E} \int_{t_1}^T \left\langle S(\tau-t_1) \int_{t_2}^{t_1} S(t_1-s) \gamma ds, f(\tau) \right\rangle_H d\tau = \mathbb{E} \int_{t_2}^T \langle S(\tau-t_2) \gamma, f(\tau) \rangle_H d\tau. \quad (3.28)$$

From (3.25), (3.26), (3.27) and (3.28), we arrive at

$$\lim_{t_1 \rightarrow t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \gamma, y(\tau) \rangle_H d\tau = \mathbb{E} \langle \gamma, \xi^{t_2} \rangle_H, \quad \forall \gamma \in L^2_{\mathcal{F}_{t_2}}(\Omega; H), \quad t_2 \in [0, T]. \quad (3.29)$$

Now, by (3.29), we conclude that, for a.e. $t_2 \in (0, T)$

$$\lim_{t_1 \rightarrow t_2+0} \frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \mathbb{E} \langle \xi^{t_2} - y(t_2), y(\tau) \rangle_H d\tau = \mathbb{E} \langle \xi^{t_2} - y(t_2), \xi^{t_2} \rangle_H. \quad (3.30)$$

By Lemma 2.5, we can find a monotonic sequence $\{h_n\}_{n=1}^{\infty}$ of positive numbers with $\lim_{n \rightarrow \infty} h_n = 0$, such that

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \int_{t_2}^{t_2+h_n} \mathbb{E} \langle \xi^{t_2} - y(t_2), y(\tau) \rangle_H d\tau = \mathbb{E} \langle \xi^{t_2} - y(t_2), y(t_2) \rangle_H, \quad \text{a.e. } t_2 \in [0, T]. \quad (3.31)$$

By (3.30)–(3.31), we arrive at

$$\mathbb{E}\langle \xi^{t_2} - y(t_2), \xi^{t_2} \rangle_H = \mathbb{E}\langle \xi^{t_2} - y(t_2), y(t_2) \rangle_H, \quad \text{a.e. } t_2 \in [0, T]. \quad (3.32)$$

By (3.32), we find that $\mathbb{E}|\xi^{t_2} - y(t_2)|_H^2 = 0$ for $t_2 \in [0, T]$ a.e., which implies (3.22) for $p = 2$ immediately.

When $p \in (1, 2]$, we choose $\{y_T^n\}_{n=1}^\infty \subset L_{\mathcal{F}_T}^2(\Omega; H)$ and $\{f_n\}_{n=1}^\infty \subset L_{\mathbb{F}}^1(0, T; L^2(\Omega; H))$ such that

$$\begin{cases} \lim_{n \rightarrow \infty} y_T^n = y_T \text{ in } L_{\mathcal{F}_T}^p(\Omega; H), \\ \lim_{n \rightarrow \infty} f_n = f \text{ in } L_{\mathbb{F}}^1(0, T; L^p(\Omega; H)). \end{cases} \quad (3.33)$$

We replace y_T (resp. f) by y_T^n (resp. f_n) in the definition of the functional ℓ (See (3.3)) and denote by $(y_n(\cdot), Y_n(\cdot), \xi_n^t)$ the corresponding triple satisfying (3.12). By the definition of $(y(\cdot), Y(\cdot), \xi^t)$ and $(y_n(\cdot), Y_n(\cdot), \xi_n^t)$, it is easy to see that $(y(\cdot) - y_n(\cdot), Y(\cdot) - Y_n(\cdot))$, $n = 1, 2, \dots$, satisfy the following:

$$\begin{aligned} & \mathbb{E}\langle z(T), y_T - y_T^n \rangle_H - \mathbb{E}\langle \eta, \xi^t - \xi_n^t \rangle_H \\ &= \mathbb{E} \int_t^T \langle z(\tau), f(\tau) - f_n(\tau) \rangle_H d\tau + \mathbb{E} \int_t^T \langle v_1(\tau), y(\tau) - y_n(\tau) \rangle_H d\tau \\ & \quad + \mathbb{E} \int_t^T \langle v_2(\tau), Y(\tau) - Y_n(\tau) \rangle_H d\tau, \\ & \quad \forall (v_1(\cdot), v_2(\cdot), \eta) \in L_{\mathbb{F}}^1(t, T; L^q(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^q(\Omega; H)) \times L_{\mathcal{F}_t}^q(\Omega; H). \end{aligned} \quad (3.34)$$

Hence,

$$\begin{aligned} & |y(\cdot) - y_n(\cdot)|_{L_{\mathbb{F}}^\infty(0, T; L^p(\Omega; H))} + |\xi^t - \xi_n^t|_{L_{\mathcal{F}_t}^p(\Omega; H)} \\ & \leq C(|f - f_n|_{L_{\mathbb{F}}^1(0, T; L^p(\Omega; H))} + |y_T - y_T^n|_{L_{\mathcal{F}_T}^p(\Omega; H)}). \end{aligned} \quad (3.35)$$

Here the constant C is independent of n . From the above inequality, we conclude

$$\lim_{n \rightarrow \infty} y_n(\cdot) = y(\cdot) \text{ in } L_{\mathbb{F}}^\infty(0, T; L^p(\Omega; H)) \text{ and } \lim_{n \rightarrow \infty} \xi_n^t = \xi^t \text{ in } L_{\mathcal{F}_t}^p(\Omega; H).$$

Therefore,

$$|y(t) - \xi^t|_{L_{\mathcal{F}_t}^p(\Omega; H)} \leq \lim_{n \rightarrow \infty} |y_n(t) - \xi_n^t|_{L_{\mathcal{F}_t}^p(\Omega; H)} \leq \lim_{n \rightarrow \infty} |y_n(t) - \xi_n^t|_{L_{\mathcal{F}_t}^2(\Omega; H)} = 0, \quad \text{a.e. } t \in [0, T],$$

which implies (3.22) immediately.

Finally, by (3.22) and recalling that ξ^t has a càdlàg modification, we see that there is a càdlàg H -valued process $\{\tilde{y}(t)\}_{t \in [0, T]}$ such that $y(\cdot) = \tilde{y}(\cdot)$ in $[0, T] \times \Omega$ a.e. It is easy to check that $(\tilde{y}(\cdot), Y(\cdot))$ is a transposition solution to the equation (3.2). To simplify the notation, we still use y (instead of \tilde{y}) to denote the first component of the solution. Clearly, $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(0, T; L^p(\Omega; H))$ satisfies that

$$\begin{aligned} & |(y(\cdot), Y(\cdot))|_{L_{\mathbb{F}}^\infty(t, T; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^p(\Omega; H))} \\ & \leq C \left[|f(\cdot)|_{L_{\mathbb{F}}^1(t, T; L^p(\Omega; H))} + |y_T|_{L_{\mathcal{F}_T}^p(\Omega; H)} \right], \quad \forall t \in [0, T]. \end{aligned} \quad (3.36)$$

Also, the uniqueness of the transposition solution to (3.2) is obvious.

Step 5. In this step, we consider the equation (1.8) for the general case.

Fix any $T_1 \in [0, T]$. For any $(\sigma(\cdot), \Sigma(\cdot)) \in D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))$, we consider the following equation:

$$\begin{cases} dy_1 = -A^*y_1dt + f(t, \sigma(t), \Sigma(t))dt + Y_1dw(t) & \text{in } [T_1, T], \\ y_1(T) = y_T. \end{cases} \quad (3.37)$$

By the condition (1.9) and the result obtained in the above, the equation (3.37) admits a unique transposition solution $(y_1(\cdot), Y_1(\cdot)) \in D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))$. This defines a map \mathcal{J} as

$$\begin{cases} \mathcal{J} : D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H)) \rightarrow D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H)), \\ \mathcal{J}(p(\cdot), P(\cdot)) = (y_1(\cdot), Y_1(\cdot)). \end{cases}$$

Now we show that the map \mathcal{J} is contractive provided that $T - T_1$ is small enough. Indeed, for another $(\theta(\cdot), \Theta(\cdot)) \in D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))$, we define $(y_2(\cdot), Y_2(\cdot)) = \mathcal{J}(\theta(\cdot), \Theta(\cdot))$. Put

$$y_3(\cdot) = y_1(\cdot) - y_2(\cdot), \quad Y_3(\cdot) = Y_1(\cdot) - Y_2(\cdot), \quad f_3(\cdot) = f(\cdot, p(\cdot), P(\cdot)) - f(\cdot, q(\cdot), Q(\cdot)).$$

Clearly, $(y_3(\cdot), Y_3(\cdot))$ solves the following equation

$$\begin{cases} dy_3 = -A^*y_3dt + f_3(t)dt + Y_3dw(t) & \text{in } [T_1, T], \\ y_3(T) = 0. \end{cases} \quad (3.38)$$

By the condition (1.9), it is easy to see that $f_3(\cdot) \in L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))$ and

$$\begin{aligned} & |f_3(\cdot)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} \\ & \leq C_L \left[|\sigma(\cdot) - \theta(\cdot)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + |\Sigma(\cdot) - \Theta(\cdot)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} \right] \\ & \leq C_L (T - T_1 + \sqrt{T - T_1}) \left[|\sigma(\cdot) - \theta(\cdot)|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H))} + |\Sigma(\cdot) - \Theta(\cdot)|_{L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \right]. \end{aligned} \quad (3.39)$$

By (3.36), it follows that

$$\begin{aligned} & |(y_3(\cdot), Y_3(\cdot))|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \leq C |f_3(\cdot)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} \\ & \leq C (T - T_1 + \sqrt{T - T_1}) \left[|\sigma(\cdot) - \theta(\cdot)|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H))} + |\Sigma(\cdot) - \Theta(\cdot)|_{L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \right]. \end{aligned} \quad (3.40)$$

Choose T_1 so that $C(T - T_1 + \sqrt{T - T_1}) < 1$. Then, \mathcal{J} is a contractive map.

By means of the Banach fixed point theorem, \mathcal{J} enjoys a unique fixed point $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))$. It is clear that $(y(\cdot), Y(\cdot))$ is a transposition solution to the following equation:

$$\begin{cases} dy(t) = -A^*y(t)dt + f(t, y(t), Y(t))dt + Y(t)dw(t) & \text{in } [T_1, T], \\ y(T) = y_T. \end{cases} \quad (3.41)$$

Using again (1.9) and similar to (3.39), we see that $f(\cdot, y(\cdot), Y(\cdot)) \in L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))$ and

$$\begin{aligned} & |f(\cdot, y(\cdot), Y(\cdot))|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} \\ & \leq |f(\cdot, 0, 0)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + C_L \left[|y(\cdot)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + |Y(\cdot)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} \right] \\ & \leq |f(\cdot, 0, 0)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + C_L (T - T_1 + \sqrt{T - T_1}) \left[|y(\cdot)|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H))} + |Y(\cdot)|_{L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \right]. \end{aligned} \quad (3.42)$$

Therefore, we find that

$$\begin{aligned}
& |(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \\
& \leq C \left[|f(\cdot, y(\cdot), Y(\cdot))|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + |y_T|_{L_{\mathcal{F}_T}^p(\Omega; H)} \right] \\
& \leq C \left[(T - T_1 + \sqrt{T - T_1}) |(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \right. \\
& \quad \left. + |f(\cdot, 0, 0)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + |y_T|_{L_{\mathcal{F}_T}^p(\Omega; H)} \right].
\end{aligned} \tag{3.43}$$

Since $C(T - T_1 + \sqrt{T - T_1}) < 1$, it follows from (3.43) that

$$|(y(\cdot), Y(\cdot))|_{D_{\mathbb{F}}([T_1, T]; L^p(\Omega; H)) \times L_{\mathbb{F}}^2(T_1, T; L^p(\Omega; H))} \leq C [|f(\cdot, 0, 0)|_{L_{\mathbb{F}}^1(T_1, T; L^p(\Omega; H))} + |y_T|_{L_{\mathcal{F}_T}^p(\Omega; H)}]. \tag{3.44}$$

Repeating the above argument, we obtain the transposition solution of the equation (1.8). The uniqueness of such solution to (1.8) is obvious. The desired estimate (3.1) follows from (3.44). This completes the proof of Theorem 3.1. \square

4 Well-posedness result for the operator-valued BSEEs with special data

This section is addressed to proving a well-posedness result for the transposition solutions of the operator-valued BSEEs with special data P_T and F .

We begin with the following uniqueness result for the transposition solution to (1.10).

Theorem 4.1 *If $P_T \in L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))$, $F \in L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))$ and $J, K \in L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, then (1.10) admits at most one transposition solution $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F}, w}([0, T]; L^2(\Omega; \mathcal{L}(H))) \times L_{\mathbb{F}, w}^2(0, T; L^2(\Omega; \mathcal{L}(H)))$.*

Proof: Assume that $(\bar{P}(\cdot), \bar{Q}(\cdot))$ is another transposition solution to the equation (1.10). Then, by Definition 1.2, it follows that

$$\begin{aligned}
0 &= \mathbb{E} \left\langle (\bar{P}(t) - P(t)) \xi_1, \xi_2 \right\rangle_H + \mathbb{E} \int_t^T \left\langle (\bar{P}(s) - P(s)) u_1(s), x_2(s) \right\rangle_H ds \\
&+ \mathbb{E} \int_t^T \left\langle (\bar{P}(s) - P(s)) x_1(s), u_2(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle (\bar{P}(s) - P(s)) K(s) x_1(s), v_2(s) \right\rangle_H ds \\
&+ \mathbb{E} \int_t^T \left\langle (\bar{P}(s) - P(s)) v_1(s), K(s) x_2(s) + v_2(s) \right\rangle_H ds \\
&+ \mathbb{E} \int_t^T \left\langle (\bar{Q}(s) - Q(s)) v_1(s), x_2(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle (\bar{Q}(s) - Q(s)) x_1(s), v_2(s) \right\rangle_H ds, \\
&\quad \forall t \in [0, T].
\end{aligned} \tag{4.1}$$

Choosing $u_1 = v_1 = 0$ and $u_2 = v_2 = 0$ in equations (1.13) and (1.14), respectively, by (4.1), we obtain that, for any $t \in [0, T]$,

$$0 = \mathbb{E} \left\langle (\bar{P}(t) - P(t)) \xi_1, \xi_2 \right\rangle_H, \quad \forall \xi_1, \xi_2 \in L_{\mathcal{F}_t}^4(\Omega; H).$$

Hence, we find that $\overline{P}(\cdot) = P(\cdot)$. By this, it is easy to see that (4.1) becomes the following

$$0 = \mathbb{E} \int_t^T \left\langle \left(\overline{Q}(s) - Q(s) \right) v_1(s), x_2(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle \left(\overline{Q}(s) - Q(s) \right) x_1(s), v_2(s) \right\rangle_H ds, \quad (4.2)$$

$\forall t \in [0, T].$

Choosing $t = 0$, $\xi_2 = 0$ and $v_2 = 0$ in the equation (1.14), we see that (4.2) becomes

$$0 = \mathbb{E} \int_0^T \left\langle \left(\overline{Q}(s) - Q(s) \right) v_1(s), x_2(s) \right\rangle_H ds. \quad (4.3)$$

We claim that the set

$$\Xi \triangleq \left\{ x_2(\cdot) \mid x_2(\cdot) \text{ solves (1.14) with } t = 0, \xi_2 = 0, v_2 = 0 \text{ and some } u_2 \in L_{\mathbb{F}}^4(0, T; H) \right\}$$

is dense in $L_{\mathbb{F}}^4(0, T; H)$. Indeed, arguing by contradiction, if this was not true, then there would be a nonzero $r \in L_{\mathbb{F}}^{\frac{4}{3}}(0, T; H)$ such that

$$\mathbb{E} \int_0^T \langle r, x_2 \rangle_H ds = 0, \quad \text{for any } x_2 \in \Xi. \quad (4.4)$$

Let us consider the following H -valued BSEE:

$$\begin{cases} dy = -A^* y dt + (r - J(t)^* y - K(t)^* Y) dt + Y dw(t), & \text{in } [0, T], \\ y(T) = 0. \end{cases} \quad (4.5)$$

The solution to the equation (4.5) is understood in the transposition sense. By Theorem 3.1, the BSEE (4.5) admits one and only one transposition solution $(y(\cdot), Y(\cdot)) \in D_{\mathbb{F}}([0, T]; L_{\mathbb{F}}^{\frac{4}{3}}(\Omega; H)) \times L_{\mathbb{F}}^2(0, T; L_{\mathbb{F}}^{\frac{4}{3}}(\Omega; H))$. Hence, for any $\phi_1(\cdot) \in L_{\mathbb{F}}^1(0, T; L^4(\Omega; H))$ and $\phi_2(\cdot) \in L_{\mathbb{F}}^2(0, T; L^4(\Omega; H))$, it holds that

$$\begin{aligned} & -\mathbb{E} \int_0^T \langle z(s), r(s) - J(s)^* y(s) - K(s)^* Y(s) \rangle_H ds \\ &= \mathbb{E} \int_0^T \langle \phi_1(s), y(s) \rangle_H ds + \mathbb{E} \int_0^T \langle \phi_2(s), Y(s) \rangle_H ds, \end{aligned} \quad (4.6)$$

where $z(\cdot)$ solves

$$\begin{cases} dz = (Az + \phi_1) dt + \phi_2 dw(t), & \text{in } (0, T], \\ z(0) = 0. \end{cases} \quad (4.7)$$

In particular, for any $x_2(\cdot)$ solving (1.14) with $t = 0$, $\xi_2 = 0$, $v_2 = 0$ and an arbitrarily given $u_2 \in L_{\mathbb{F}}^4(0, T; H)$, we choose $z = x_2$, $\phi_1 = Jx_2 + u_2$ and $\phi_2 = Kx_2$. By (4.6), it follows that

$$-\mathbb{E} \int_0^T \langle x_2(s), r(s) \rangle_H ds = \mathbb{E} \int_0^T \langle u_2(s), y(s) \rangle_H ds, \quad \forall u_2 \in L_{\mathbb{F}}^4(0, T; H). \quad (4.8)$$

By (4.8) and recalling (4.4), we conclude that $y(\cdot) = 0$. Hence, (4.6) is reduced to

$$-\mathbb{E} \int_0^T \langle z(s), r(s) - K(s)^* Y(s) \rangle_H ds = \mathbb{E} \int_0^T \langle \phi_2(s), Y(s) \rangle_H ds. \quad (4.9)$$

Choosing $\phi_2(\cdot) = 0$ in (4.7) and (4.9), we obtain that

$$\mathbb{E} \int_0^T \left\langle \int_0^s S(s-\sigma) \phi_1(\sigma) d\sigma, r(s) - K(s)^* Y(s) \right\rangle_H ds = 0, \quad \forall \phi_1(\cdot) \in L_{\mathbb{F}}^1(0, T; L^4(\Omega; H)). \quad (4.10)$$

Hence,

$$\int_{\sigma}^T S(s-\sigma) [r(s) - K(s)^* Y(s)] ds = 0, \quad \forall \sigma \in [0, T]. \quad (4.11)$$

Then, for any given $\lambda_0 \in \rho(A)$ and $\sigma \in [0, T]$, we have

$$\begin{aligned} & \int_{\sigma}^T S(s-\sigma) (\lambda_0 - A)^{-1} [r(s) - K(s)^* Y(s)] ds \\ &= (\lambda_0 - A)^{-1} \int_{\sigma}^T S(s-\sigma) [r(s) - K(s)^* Y(s)] ds = 0. \end{aligned} \quad (4.12)$$

Differentiating the equality (4.12) with respect to σ , and noting (4.11), we see that

$$\begin{aligned} (\lambda_0 - A)^{-1} [r(\sigma) - K(\sigma)^* Y(\sigma)] &= - \int_{\sigma}^T S(s-\sigma) A (\lambda_0 - A)^{-1} [r(s) - K(s)^* Y(s)] ds \\ &= \int_{\sigma}^T S(s-\sigma) [r(s) - K(s)^* Y(s)] ds \\ &\quad - \lambda_0 \int_{\sigma}^T S(s-\sigma) (\lambda_0 - A)^{-1} [r(s) - K(s)^* Y(s)] ds \\ &= 0, \quad \forall \sigma \in [0, T]. \end{aligned}$$

Therefore,

$$r(\cdot) = K(\cdot)^* Y(\cdot). \quad (4.13)$$

By (4.13), the equation (4.5) is reduced to

$$\begin{cases} dy = -A^* y dt - J(s)^* y dt + Y dw(t), & \text{in } [0, T], \\ y(T) = 0. \end{cases} \quad (4.14)$$

It is clear that the unique transposition of (4.14) is $(y(\cdot), Y(\cdot)) = (0, 0)$. Hence, by (4.13), we conclude that $r(\cdot) = 0$, which is a contradiction. Therefore, Ξ is dense in $L_{\mathbb{F}}^4(0, T; H)$. This, combined with (4.3), yields that

$$(\overline{Q}(\cdot) - Q(\cdot)) v_1(\cdot) = 0, \quad \forall v_1(\cdot) \in L_{\mathbb{F}}^4(0, T; H).$$

Hence $\overline{Q}(\cdot) = Q(\cdot)$. This completes the proof of Theorem 4.1. \square

In the rest of this section, we assume that H is a separable Hilbert space. Denote by $\mathcal{L}_2(H)$ the Hilbert space of all Hilbert-Schmidt operators on H . We have the following well-posedness result.

Theorem 4.2 *If $P_T \in L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}_2(H))$, $F \in L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}_2(H)))$ and $J, K \in L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, then the equation (1.10) admits one and only one transposition solution $(P(\cdot), Q(\cdot))$ with the following regularity:*

$$(P(\cdot), Q(\cdot)) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}_2(H))) \times L_{\mathbb{F}}^2(0, T; \mathcal{L}_2(H)).$$

Furthermore,

$$|(P, Q)|_{D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}_2(H))) \times L_{\mathbb{F}}^2(0, T; \mathcal{L}_2(H))} \leq C \left[|F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}_2(H)))} + |P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}_2(H))} \right]. \quad (4.15)$$

Proof: We divide the proof into several steps.

Step 1. Define a family of operators $\{\mathcal{T}(t)\}_{t \geq 0}$ on $\mathcal{L}_2(H)$ as follows:

$$\mathcal{T}(t)O = S(t)OS^*(t), \quad \forall O \in \mathcal{L}_2(H).$$

We claim that $\{\mathcal{T}(t)\}_{t \geq 0}$ is a C_0 -semigroup on $\mathcal{L}_2(H)$. Indeed, for any nonnegative s and t , we have

$$\mathcal{T}(t+s)O = S(t+s)OS^*(t+s) = S(t)S(s)OS^*(s)S^*(t) = \mathcal{T}(t)\mathcal{T}(s)O, \quad \forall O \in \mathcal{L}_2(H).$$

Hence, $\{\mathcal{T}(t)\}_{t \geq 0}$ is a semigroup on $\mathcal{L}_2(H)$. Next, we choose an orthonormal basis $\{e_i\}_{i=1}^\infty$ of H . For any $O \in \mathcal{L}_2(H)$ and $t \in [0, \infty)$,

$$\begin{aligned} & \lim_{s \rightarrow t^+} \|\mathcal{T}(s)O - \mathcal{T}(t)O\|_{\mathcal{L}_2(H)}^2 \\ & \leq \|S(t)\|_{\mathcal{L}(H)}^2 \lim_{s \rightarrow t^+} \|S(s-t)OS^*(s-t) - O\|_{\mathcal{L}_2(H)}^2 \|S^*(t)\|_{\mathcal{L}(H)}^2 \\ & \leq \|S(t)\|_{\mathcal{L}(H)}^4 \lim_{s \rightarrow t^+} \sum_{i=1}^\infty |S(s-t)OS^*(s-t)e_i - Oe_i|_H^2 \\ & \leq 2\|S(t)\|_{\mathcal{L}(H)}^4 \lim_{s \rightarrow t^+} \sum_{i=1}^\infty \left[|S(s-t)OS^*(s-t)e_i - S(s-t)Oe_i|_H^2 + |S(s-t)Oe_i - Oe_i|_H^2 \right]. \end{aligned} \quad (4.16)$$

For the first series in the right hand side of (4.16), we have

$$\begin{aligned} & \sum_{i=1}^\infty |S(s-t)OS^*(s-t)e_i - S(s-t)Oe_i|_H^2 \\ & \leq C \sum_{i=1}^\infty |OS^*(s-t)e_i - Oe_i|_H^2 = C \|OS^*(s-t) - O\|_{\mathcal{L}_2(H)}^2 = C \|[OS^*(s-t) - O]^*\|_{\mathcal{L}_2(H)}^2 \\ & = C \sum_{i=1}^\infty |[S(s-t)O^* - O^*]e_i|_H^2. \end{aligned}$$

For each $i \in \mathbb{N}$,

$$|[S(s-t)O^* - O^*]e_i|_H^2 \leq 2|S(s-t)O^*e_i|_H^2 + |O^*e_i|_H^2 \leq C|O^*e_i|_H^2.$$

It is clear that

$$\sum_{i=1}^\infty |O^*e_i|_H^2 = |O^*|_{\mathcal{L}_2(H)}^2 = |O|_{\mathcal{L}_2(H)}^2.$$

Hence, by Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned} & \lim_{s \rightarrow t^+} \sum_{i=1}^\infty |S(s-t)OS^*(s-t)e_i - S(s-t)Oe_i|_H^2 \\ & \leq C \lim_{s \rightarrow t^+} \sum_{i=1}^\infty |OS^*(s-t)e_i - Oe_i|_H^2 = C \sum_{i=1}^\infty \lim_{s \rightarrow t^+} |OS^*(s-t)e_i - Oe_i|_H^2 = 0. \end{aligned} \quad (4.17)$$

By a similar argument, it follows that

$$\lim_{s \rightarrow t^+} \sum_{i=1}^\infty |S(s-t)Oe_i - Oe_i|_H^2 = 0. \quad (4.18)$$

From (4.16)–(4.18), we find that

$$\lim_{s \rightarrow t^+} |\mathcal{T}(s)O - \mathcal{T}(t)O|_{\mathcal{L}_2(H)}^2 = 0, \quad \forall t \in [0, \infty) \text{ and } O \in \mathcal{L}_2(H).$$

Similarly,

$$\lim_{s \rightarrow t^-} |\mathcal{T}(s)O - \mathcal{T}(t)O|_{\mathcal{L}_2(H)}^2 = 0, \quad \forall t \in (0, \infty) \text{ and } O \in \mathcal{L}_2(H).$$

Hence, $\{\mathcal{T}(t)\}_{t \geq 0}$ is a C_0 -semigroup on $\mathcal{L}_2(H)$.

Step 2. Denote by \mathcal{A} the infinitesimal generator of $\{\mathcal{T}(t)\}_{t \geq 0}$. We consider the following $\mathcal{L}_2(H)$ -valued BSEE:

$$\begin{cases} dP = -\mathcal{A}^* P dt + f(t, P, Q) dt + Q dw & \text{in } [0, T], \\ P(T) = P_T, \end{cases} \quad (4.19)$$

where

$$f(t, P, Q) = -J^* P - P J - K^* P K - K^* Q - Q K + F. \quad (4.20)$$

Noting that $J, K \in L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))$ and $F \in L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}_2(H)))$, we see that $f(\cdot, \cdot, \cdot)$ satisfies (1.9). By Theorem 3.1, we conclude that there exists a $(P, Q) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}_2(H))) \times L_{\mathbb{F}}^2(0, T; \mathcal{L}_2(H))$ solves (4.19) in the sense of Definition 1.1, where the Hilbert space H is replaced by $\mathcal{L}_2(H)$. Further, (P, Q) satisfies (4.15).

Denote by $O(\cdot)$ the tensor product of $x_1(\cdot)$ and $x_2(\cdot)$, where x_1 and x_2 solve respectively (1.13) and (1.14). As usual, $O(t, \omega)x = \langle x, x_1 \rangle_H x_2$ for a.e. $(t, \omega) \in [0, T] \times \Omega$ and $x \in H$. Hence, $O(t, \omega) \in \mathcal{L}_2(H)$. For any $\lambda \in \rho(A)$, define a family of operators $\{\mathcal{T}_\lambda(t)\}_{t \geq 0}$ on $\mathcal{L}_2(H)$ as follows:

$$\mathcal{T}_\lambda(t)O = S_\lambda(t)OS_\lambda^*(t), \quad \forall O \in \mathcal{L}_2(H).$$

By the result proved in Step 1, it follows that $\{\mathcal{T}_\lambda(t)\}_{t \geq 0}$ is a C_0 -semigroup on $\mathcal{L}_2(H)$. Further, for any $O \in \mathcal{L}_2(H)$, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{\mathcal{T}_\lambda(t)O - O}{t} &= \lim_{t \rightarrow 0^+} \frac{S_\lambda(t)OS_\lambda^*(t) - O}{t} = \lim_{t \rightarrow 0^+} \frac{S_\lambda(t)OS_\lambda^*(t) - OS_\lambda^*(t) + OS_\lambda^*(t) - O}{t} \\ &= A_\lambda O + OA_\lambda^*. \end{aligned}$$

Hence, the infinitesimal generator \mathcal{A}_λ of $\{\mathcal{T}_\lambda(t)\}_{t \geq 0}$ is as follows:

$$\mathcal{A}_\lambda O = A_\lambda O + OA_\lambda^*, \quad \text{for every } O \in \mathcal{L}_2(H).$$

Now, for any $O \in \mathcal{L}_2(H)$, it holds that

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} \left| \mathcal{T}(t)O - \mathcal{T}_\lambda(t)O \right|_{\mathcal{L}_2(H)} \\ &= \lim_{\lambda \rightarrow \infty} \left| S(t)OS^*(t) - S_\lambda(t)OS_\lambda^*(t) \right|_{\mathcal{L}_2(H)} \\ &\leq \lim_{\lambda \rightarrow \infty} \left| S(t)OS^*(t) - S(t)OS_\lambda^*(t) \right|_{\mathcal{L}_2(H)} + \lim_{\lambda \rightarrow \infty} \left| S(t)OS_\lambda^*(t) - S_\lambda(t)OS_\lambda^*(t) \right|_{\mathcal{L}_2(H)}. \end{aligned}$$

Let us compute the value of each term in the right hand side of the above inequality. First,

$$\begin{aligned} \left| S(t)OS^*(t) - S(t)OS_\lambda^*(t) \right|_{\mathcal{L}_2(H)}^2 &\leq C \left| OS^*(t) - OS_\lambda^*(t) \right|_{\mathcal{L}_2(H)}^2 = C \left| S(t)O^* - S_\lambda(t)O^* \right|_{\mathcal{L}_2(H)}^2 \\ &= C \sum_{i=1}^{\infty} \left| (S(t) - S_\lambda(t))O^* e_i \right|_H^2. \end{aligned} \quad (4.21)$$

Since

$$\left| \left(S(t) - S_\lambda(t) \right) O^* e_i \right|_H^2 \leq C |O^* e_i|_H^2 \quad \text{and} \quad \sum_{i=1}^{\infty} \left| O^* e_i \right|_H^2 = |O|_{\mathcal{L}_2(H)}^2 < \infty,$$

by means of Lebesgue's dominated theorem and (4.21), we find that

$$\lim_{\lambda \rightarrow \infty} \left| S(t) O S^*(t) - S(t) O S_\lambda^*(t) \right|_{\mathcal{L}_2(H)}^2 = 0.$$

By a similar argument, we find that

$$\lim_{\lambda \rightarrow \infty} \left| S(t) O S_\lambda^*(t) - S_\lambda(t) O S_\lambda^*(t) \right|_{\mathcal{L}_2(H)}^2 = 0.$$

Hence,

$$\lim_{\lambda \rightarrow \infty} \left| \mathcal{T}(t) O - \mathcal{T}_\lambda(t) O \right|_{\mathcal{L}_2(H)} = 0, \quad \text{for any } t \geq 0. \quad (4.22)$$

Write $O^\lambda = x_1^\lambda \otimes x_2^\lambda$, where x_1^λ and x_2^λ solve accordingly (2.11) and (2.12). Then,

$$dO^\lambda = (A_\lambda x_1^\lambda) \otimes x_2^\lambda ds + x_1^\lambda \otimes (A_\lambda x_2^\lambda) ds + u^\lambda ds + v^\lambda dw, \quad (4.23)$$

where

$$\begin{cases} u^\lambda = (Jx_1^\lambda) \otimes x_2^\lambda + x_1^\lambda \otimes (Jx_2^\lambda) + u_1 \otimes x_2^\lambda + x_1^\lambda \otimes u_2 + (Kx_1^\lambda) \otimes (Kx_2^\lambda) + (Kx_1^\lambda) \otimes v_2 \\ \quad + v_1 \otimes (Kx_2^\lambda) + v_1 \otimes v_2, \\ v^\lambda = (Kx_1^\lambda) \otimes x_2^\lambda + x_1^\lambda \otimes (Kx_2^\lambda) + v_1 \otimes x_2^\lambda + x_1^\lambda \otimes v_2. \end{cases} \quad (4.24)$$

Further, for any $h \in H$, we find

$$((A_\lambda x_1^\lambda) \otimes x_2^\lambda)(h) = \langle h, A_\lambda x_1^\lambda \rangle_H x_2^\lambda = \langle A_\lambda^* h, x_1^\lambda \rangle_H x_2^\lambda = (x_1^\lambda \otimes x_2^\lambda) A_\lambda^* h.$$

Thus,

$$(A_\lambda x_1^\lambda) \otimes x_2^\lambda = O^\lambda A_\lambda^*. \quad (4.25)$$

Similarly, we have the following equalities:

$$\begin{cases} x_1^\lambda \otimes (A_\lambda x_2^\lambda) = A_\lambda O^\lambda, \\ (Jx_1^\lambda) \otimes x_2^\lambda + x_1^\lambda \otimes (Jx_2^\lambda) = O^\lambda J^* + J O^\lambda, \\ (Kx_1^\lambda) \otimes (Kx_2^\lambda) = K O^\lambda K^*, \\ (Kx_1^\lambda) \otimes v_2 + v_1 \otimes (Kx_2^\lambda) = (x_1^\lambda \otimes v_2) K^* + K^* (v_1 \otimes x_2^\lambda), \\ (Kx_1^\lambda) \otimes x_2^\lambda + x_1^\lambda \otimes (Kx_2^\lambda) = O^\lambda K^* + K O^\lambda. \end{cases} \quad (4.26)$$

By (4.24)–(4.26), we obtain that

$$\begin{cases} u^\lambda = J O^\lambda + O^\lambda J^* + u_1 \otimes x_2^\lambda + x_1^\lambda \otimes u_2 + K O^\lambda K^* + K x_1^\lambda \otimes v_2 + v_1 \otimes K x_2^\lambda + v_1 \otimes v_2, \\ v^\lambda = K O^\lambda + O^\lambda K^* + v_1 \otimes x_2^\lambda + x_1^\lambda \otimes v_2. \end{cases} \quad (4.27)$$

From (4.23), (4.25), the first equality in (4.26) and (4.27), we see that O^λ solves

$$\begin{cases} dO^\lambda = \mathcal{A}_\lambda O^\lambda ds + u^\lambda ds + v^\lambda dw(s) & \text{in } (t, T], \\ O^\lambda(t) = \xi_1 \otimes \xi_2. \end{cases} \quad (4.28)$$

Hence,

$$O^\lambda(s) = \mathcal{T}_\lambda(s-t)(\xi_1 \otimes \xi_2) + \int_t^s \mathcal{T}_\lambda(\tau-t)u^\lambda(\tau)d\tau + \int_t^s \mathcal{T}_\lambda(\tau-t)v^\lambda(\tau)dw(\tau), \quad \forall s \in [t, T]. \quad (4.29)$$

We claim that

$$\lim_{\lambda \rightarrow \infty} |O^\lambda(\cdot) - O(\cdot)|_{C_{\mathbb{F}}([t, T]; L^4(\Omega; \mathcal{L}_2(H)))} = 0, \quad \forall t \in [0, T]. \quad (4.30)$$

Indeed, for any $s \in [t, T]$, we have

$$\begin{aligned} |O^\lambda(s) - O(s)|_{\mathcal{L}_2(H)}^2 &= \sum_{i=1}^{\infty} |O^\lambda(s)e_i - O(s)e_i|_H^2 \\ &= \sum_{i=1}^{\infty} |\langle e_i, x_1^\lambda(s) \rangle_H x_2^\lambda(s) - \langle e_i, x_1(s) \rangle_H x_2(s)|_H^2 \\ &\leq 2 \sum_{i=1}^{\infty} |\langle e_i, x_1^\lambda(s) \rangle_H x_2^\lambda(s) - \langle e_i, x_1^\lambda(s) \rangle_H x_2(s)|_H^2 + 2 \sum_{i=1}^{\infty} |\langle e_i, x_1^\lambda(s) \rangle_H x_2(s) - \langle e_i, x_1(s) \rangle_H x_2(s)|_H^2 \\ &\leq 2|x_2^\lambda(s) - x_2(s)|_H^2 \sum_{i=1}^{\infty} |\langle e_i, x_1^\lambda(s) \rangle_H|^2 + 2|x_2(s)|_H^2 \sum_{i=1}^{\infty} |\langle e_i, x_1^\lambda(s) - x_1(s) \rangle_H|^2 \\ &= 2|x_1(s)|_H^2 |x_2^\lambda(s) - x_2(s)|_H^2 + 2|x_2(s)|_H^2 |x_1^\lambda(s) - x_1(s)|_H^2. \end{aligned}$$

This, together Lemma 2.7, implies that (4.30) holds.

By a similar argument and noting (4.22), using Lemma 2.1 and (4.30), we can show that, for any $t \in [0, T]$, it holds that

$$\begin{cases} \lim_{\lambda \rightarrow \infty} \left| \int_t^\cdot \mathcal{T}_\lambda(\tau-t)u^\lambda(\tau)d\tau - \int_t^\cdot \mathcal{T}(\tau-t)u(\tau)d\tau \right|_{C_{\mathbb{F}}([t, T]; L^4(\Omega; \mathcal{L}_2(H)))} = 0, \\ \lim_{\lambda \rightarrow \infty} \left| \int_t^\cdot \mathcal{T}_\lambda(\tau-t)v^\lambda(\tau)dw(\tau) - \int_t^\cdot \mathcal{T}(\tau-t)v(\tau)dw(\tau) \right|_{C_{\mathbb{F}}([t, T]; L^4(\Omega; \mathcal{L}_2(H)))} = 0, \end{cases} \quad (4.31)$$

where

$$\begin{cases} u = JO(\cdot) + O(\cdot)J^* + u_1 \otimes x_2 + x_1 \otimes u_2 + KO(\cdot)K^* + (Kx_1) \otimes v_2 + v_1 \otimes (Kx_2) + v_1 \otimes v_2, \\ v = KO(\cdot) + O(\cdot)K^* + v_1 \otimes x_2 + x_1 \otimes v_2. \end{cases} \quad (4.32)$$

From (4.29)–(4.31), we obtain that

$$O(s) = \mathcal{T}(s-t)(\xi_1 \otimes \xi_2) + \int_t^s \mathcal{T}(\tau-t)u(\tau)d\tau + \int_t^s \mathcal{T}(\tau-t)v(\tau)dw(\tau), \quad \forall s \in [t, T]. \quad (4.33)$$

Hence, $O(\cdot)$ verifies that

$$\begin{cases} dO(s) = \mathcal{A}O(s)ds + uds + vdw(s) & \text{in } (t, T], \\ O(t) = \xi_1 \otimes \xi_2. \end{cases} \quad (4.34)$$

Step 3. Since (P, Q) solves (4.19) in the transposition sense and by (4.34), it follows that

$$\begin{aligned} &\mathbb{E} \langle O(T), P_T \rangle_{\mathcal{L}_2(H)} - \mathbb{E} \int_t^T \langle O(s), f(s, P(s), Q(s)) \rangle_{\mathcal{L}_2(H)} ds \\ &= \mathbb{E} \langle \xi_1 \otimes \xi_2, P(t) \rangle_{\mathcal{L}_2(H)} + \mathbb{E} \int_t^T \langle u(s), P(s) \rangle_{\mathcal{L}_2(H)} ds + \mathbb{E} \int_t^T \langle v(s), Q(s) \rangle_{\mathcal{L}_2(H)} ds. \end{aligned} \quad (4.35)$$

By (4.20) and recalling that $O(\cdot) = x_1(\cdot) \otimes x_2(\cdot)$, we find that

$$\begin{aligned} & \mathbb{E} \int_t^T \overline{\langle O(s), f(s, P(s), Q(s)) \rangle_{\mathcal{L}_2(H)}} ds \\ &= \mathbb{E} \int_t^T \langle (-J(s)^*P(s) - P(s)J(s) - K^*(s)P(s)K(s) - K^*(s)Q(s) - Q(s)K(s) \\ & \quad + F(s))x_1(s), x_2(s) \rangle_H ds. \end{aligned} \quad (4.36)$$

Further, by (4.32), we have

$$\begin{aligned} & \mathbb{E} \int_t^T \overline{\langle u(s), P(s) \rangle_{\mathcal{L}_2(H)}} ds \\ &= \mathbb{E} \int_t^T \langle J^*(s)P(s)x_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)J(s)x_1(s), x_2(s) \rangle_H ds \\ & \quad + \mathbb{E} \int_t^T \langle P(s)u_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)x_1(s), u_2(s) \rangle_H ds \\ & \quad + \mathbb{E} \int_t^T \langle K^*(s)P(s)K(s)x_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)K(s)x_1(s), v_2(s) \rangle_H ds + \\ & \quad + \mathbb{E} \int_t^T \langle K(s)^*P(s)v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)v_1(s), v_2(s) \rangle_H ds, \end{aligned} \quad (4.37)$$

and

$$\begin{aligned} & \mathbb{E} \int_t^T \overline{\langle v(s), Q(s) \rangle_{\mathcal{L}_2(H)}} ds \\ &= \mathbb{E} \int_t^T \langle K^*(s)Q(s)x_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q(s)K(s)x_1(s), x_2(s) \rangle_H ds \\ & \quad + \mathbb{E} \int_t^T \langle Q(s)v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle x_1(s), Q(s)v_2(s) \rangle_H ds. \end{aligned} \quad (4.38)$$

From (4.35)–(4.38), we see that $(P(\cdot), Q(\cdot))$ satisfies (1.17). Hence, $(P(\cdot), Q(\cdot))$ is a transposition solution of (1.10) (in the sense of Definition 1.2). The uniqueness of $(P(\cdot), Q(\cdot))$ follows from Theorem 4.1. This concludes the proof of Theorem 4.2. \square

Remark 4.1 Theorems 4.1–4.2 indicate that, in some sense, the transposition solution introduced in Definition 1.2 is a reasonable notion for the solution to (1.10). Unfortunately, we are unable to prove the existence of transposition solution to (1.10) in the general case though a weak version, i.e., the relaxed transposition solution to this equation, introduced/studied in the next three sections, suffices to establish the desired Pontryagin-type stochastic maximum principle for Problem (P) in the general setting.

5 Sequential Banach-Alaoglu-type theorems in the operator version

The classical Banach-Alaoglu Theorem (e.g. [8, p. 130]) states that the closed unit ball of the dual space of a normed vector space is compact in the weak* topology. This theorem has an important special (sequential) version, asserting that the closed unit ball of the dual space of a separable normed vector space (*resp.*, the closed unit ball of a reflexive Banach space) is sequentially compact in the weak* topology (*resp.*, the weak topology). In this section, we shall present several sequential

Banach-Alaoglu-type theorems for uniformly bounded linear operators (between suitable Banach spaces). These results will play crucial roles in the study of the well-posedness of (1.10) in the general case.

Let $\{y_n\}_{n=1}^\infty \subset Y$ and $y \in Y$. Let $\{z_n\}_{n=1}^\infty \subset Y^*$ and $z \in Y^*$. In the sequel, we denote by

$$(w)\text{-}\lim_{n \rightarrow \infty} y_n = y \quad \text{in } Y$$

when $\{y_n\}_{n=1}^\infty$ weakly converges to y in Y ; and by

$$(w^*)\text{-}\lim_{n \rightarrow \infty} z_n = z \quad \text{in } Y^*$$

when $\{z_n\}_{n=1}^\infty$ weakly* converges to z in Y^* . Let us show first the following result (It seems for us that this is a known result. However we have not found it in any reference):

Lemma 5.1 *Let X be a separable Banach space and let Y be a reflexive Banach space. Assume that $\{G_n\}_{n=1}^\infty \subset \mathcal{L}(X, Y)$ is a sequence of bounded linear operators such that $\{G_n x\}_{n=1}^\infty$ is bounded for any given $x \in X$. Then, there exist a subsequence $\{G_{n_k}\}_{k=1}^\infty$ and a bounded linear operator G from X to Y such that*

$$\begin{aligned} (w)\text{-}\lim_{k \rightarrow \infty} G_{n_k} x &= Gx \quad \text{in } Y, & \forall x \in X, \\ (w^*)\text{-}\lim_{k \rightarrow \infty} G_{n_k}^* y^* &= G^* y^* \quad \text{in } X^*, & \forall y^* \in Y^*, \end{aligned}$$

and

$$\|G\|_{\mathcal{L}(X, Y)} \leq \sup_{n \in \mathbb{N}} \|G_n\|_{\mathcal{L}(X, Y)} (< \infty). \quad (5.1)$$

Remark 5.1 *Lemma 5.1 is not a direct consequence of the classical sequential Banach-Alaoglu Theorem. Indeed, as we mentioned before, the Banach space $\mathcal{L}(X, Y)$ is neither reflexive nor separable even if both X and Y are (infinite dimensional) separable Hilbert spaces.*

Proof of Lemma 5.1: Noting that X is separable, we can find a countable subset $\{x_i\}_{i=1}^\infty$ of X such that $\{x_1, x_2, \dots\}$ is dense in X . Since $\{G_n x_1\}_{n=1}^\infty$ is bounded in Y and Y is reflexive, there exists a subsequence $\{n_k^{(1)}\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty$ such that $(w)\text{-}\lim_{k \rightarrow \infty} G_{n_k^{(1)}} x_1 = y_1$. Now, the sequence $\{G_{n_k^{(1)}} x_2\}_{k=1}^\infty$ is still bounded in Y , one can find a subsequence $\{n_k^{(2)}\}_{k=1}^\infty \subset \{n_k^{(1)}\}_{k=1}^\infty$ such that $(w)\text{-}\lim_{k \rightarrow \infty} G_{n_k^{(2)}} x_2 = y_2$. By the induction, for any $m \in \mathbb{N}$, we can find a subsequence $\{n_k^{(m+1)}\}_{k=1}^\infty \subset \{n_k^{(m)}\}_{k=1}^\infty \subset \dots \subset \{n_k^{(1)}\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty$ such that $(w)\text{-}\lim_{k \rightarrow \infty} G_{n_k^{(m+1)}} x_{m+1} = y_{m+1}$. We now use the classical diagonalisation argument. Write $n_m = n_{n_m}^{(m)}$, $m = 1, 2, \dots$. Then, it is clear that $\{G_{n_m} x_i\}_{m=1}^\infty$ converges weakly to y_i in Y .

Let us define an operator G (from X to Y) as follows: For any $x \in X$,

$$Gx = \lim_{k \rightarrow \infty} y_{i_k} = \lim_{k \rightarrow \infty} \left((w)\text{-}\lim_{m \rightarrow \infty} G_{n_m} x_{i_k} \right),$$

where $\{x_{i_k}\}_{k=1}^\infty$ is any subsequence of $\{x_i\}_{i=1}^\infty$ such that $\lim_{k \rightarrow \infty} x_{i_k} = x$ in X . We shall show below that $G \in \mathcal{L}(X, Y)$.

First, we show that G is well-defined. By the Principle of Uniform Boundedness, it is clear that $\{G_n\}_{n=1}^\infty$ is uniformly bounded in $\mathcal{L}(X, Y)$. We choose $M > 0$ such that $|G_n|_{\mathcal{L}(X, Y)} \leq M$ for all $n \in \mathbb{N}$. Since $\{x_{i_k}\}_{k=1}^\infty$ is a Cauchy sequence in X , for any $\varepsilon > 0$, there is a $N > 0$ such that

$|x_{i_{k_1}} - x_{i_{k_2}}| < \frac{\varepsilon}{M}$ when $k_1, k_2 > N$. Hence, $|G_n(x_{i_{k_1}} - x_{i_{k_2}})|_Y < \varepsilon$ for any $n \in \mathbb{N}$. Then, by the weakly sequentially lower semicontinuity (of Banach spaces), we deduce that

$$|y_{i_{k_1}} - y_{i_{k_2}}|_Y \leq \liminf_{m \rightarrow \infty} |G_{n_m}(x_{i_{k_1}} - x_{i_{k_2}})|_Y < \varepsilon,$$

which implies that $\{y_{i_k}\}_{k=1}^\infty$ is a Cauchy sequence in Y . Therefore, we see that $\lim_{k \rightarrow \infty} y_{i_k}$ exists in Y . On the other hand, assume that there is another subsequence $\{x'_{i_k}\}_{k=1}^\infty \subset \{x_i\}_{i=1}^\infty$ such that $\lim_{k \rightarrow \infty} x'_{i_k} = x$. Let y'_{i_k} be the corresponding weak limit of $G_{n_m}x'_{i_k}$ in Y for $m \rightarrow \infty$. Then we find that

$$\begin{aligned} \left| \lim_{k \rightarrow \infty} y_{i_k} - \lim_{k \rightarrow \infty} y'_{i_k} \right|_Y &\leq \lim_{k \rightarrow \infty} \liminf_{m \rightarrow \infty} |G_{n_m}(x_{i_k} - x'_{i_k})|_Y \leq M \lim_{k \rightarrow \infty} |x_{i_k} - x'_{i_k}|_X \\ &\leq M \lim_{k \rightarrow \infty} |x_{i_k} - x|_X + M \lim_{k \rightarrow \infty} |x - x'_{i_k}|_X = 0. \end{aligned}$$

Hence, G is well-defined.

Next, we prove that G is a bounded linear operator. For any $x \in X$ and the above sequence $\{x_{i_k}\}_{k=1}^\infty$, it follows that

$$|Gx|_Y = \lim_{k \rightarrow \infty} |y_{i_k}|_Y \leq \lim_{k \rightarrow \infty} \liminf_{m \rightarrow \infty} |G_{n_m}x_{i_k}|_Y \leq M \lim_{k \rightarrow \infty} |x_{i_k}|_X \leq M|x|_X.$$

Hence, G is a bounded operator. Further, for any $x^{(1)}, x^{(2)} \in X$, $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$, we choose $\{x_{i_k}^{(j)}\}_{k=1}^\infty \subset \{x_i\}_{i=1}^\infty$, $j = 1, 2$, such that $\lim_{k \rightarrow \infty} x_{i_k}^{(j)} = x^{(j)}$, and denote by $y_{i_k}^{(j)}$ the weak limit of $G_{n_m}x_{i_k}^{(j)}$ in Y for $m \rightarrow \infty$. Hence $Gx^{(j)} = \lim_{k \rightarrow \infty} y_{i_k}^{(j)}$. Then,

$$\alpha x^{(1)} + \beta x^{(2)} = \lim_{k \rightarrow \infty} (\alpha x_{i_k}^{(1)} + \beta x_{i_k}^{(2)}) = \alpha \lim_{k \rightarrow \infty} x_{i_k}^{(1)} + \beta \lim_{k \rightarrow \infty} x_{i_k}^{(2)}$$

and

$$(w)\text{-}\lim_{m \rightarrow \infty} G_{n_m}(\alpha x_{i_k}^{(1)} + \beta x_{i_k}^{(2)}) = \alpha \left((w)\text{-}\lim_{m \rightarrow \infty} G_{n_m}x_{i_k}^{(1)} \right) + \beta \left((w)\text{-}\lim_{m \rightarrow \infty} G_{n_m}x_{i_k}^{(2)} \right).$$

Hence,

$$\begin{aligned} G(\alpha x^{(1)} + \beta x^{(2)}) &= \lim_{k \rightarrow \infty} \left((w)\text{-}\lim_{m \rightarrow \infty} G_{n_m}(\alpha x_{i_k}^{(1)} + \beta x_{i_k}^{(2)}) \right) \\ &= \alpha \lim_{k \rightarrow \infty} \left((w)\text{-}\lim_{m \rightarrow \infty} G_{n_m}x_{i_k}^{(1)} \right) + \beta \lim_{k \rightarrow \infty} \left((w)\text{-}\lim_{m \rightarrow \infty} G_{n_m}x_{i_k}^{(2)} \right) \\ &= \alpha Gx^{(1)} + \beta Gx^{(2)}. \end{aligned}$$

Therefore, $G \in \mathcal{L}(X, Y)$.

Also, for any $x \in X$ and $y^* \in Y^*$, it holds that

$$(x, G^*y^*)_{X, X^*} = (Gx, y^*)_{Y, Y^*} = \lim_{k \rightarrow \infty} (G_{n_k}x, y^*)_{Y, Y^*} = \lim_{k \rightarrow \infty} (x, G_{n_k}^*y^*)_{X, X^*}.$$

Hence,

$$(w^*)\text{-}\lim_{k \rightarrow \infty} G_{n_k}^*y^* = G^*y^* \text{ in } X^*.$$

Finally, from the above proof, (5.1) is obvious. This completes the proof of Lemma 5.1. \square

Let us introduce the following set class:

$$\mathcal{M} = \left\{ O \in (0, T) \times \Omega \mid \{\chi_O(\cdot)\} \text{ is an } \mathbb{F}\text{-adapted process} \right\}. \quad (5.2)$$

This set class will be used several times in the sequel. We now show the following ‘‘stochastic process’’ version of Lemma 5.1.

Theorem 5.1 *Let X and Y be respectively a separable and a reflexive Banach space, and let $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, with $1 \leq p < \infty$, be separable. Let $1 \leq p_1, p_2 < \infty$ and $1 < q_1, q_2 < \infty$. Assume that $\{\mathcal{G}_n\}_{n=1}^\infty$ is a sequence of uniformly bounded, pointwisely defined linear operators from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. Then, there exist a subsequence $\{\mathcal{G}_{n_k}\}_{k=1}^\infty \subset \{\mathcal{G}_n\}_{n=1}^\infty$ and a $\mathcal{G} \in \mathcal{L}_{pd}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$ such that*

$$\mathcal{G}u(\cdot) = (w)\text{-}\lim_{k \rightarrow \infty} \mathcal{G}_{n_k} u(\cdot) \text{ in } L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)), \quad \forall u(\cdot) \in L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)).$$

Moreover, $\|\mathcal{G}\|_{\mathcal{L}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))} \leq \sup_{n \in \mathbb{N}} \|\mathcal{G}_n\|_{\mathcal{L}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))}$.

Remark 5.2 *i) As we shall see later, the most difficult part in the proof of Theorem 5.1 is to show that the weak limit operator \mathcal{G} is a bounded, pointwisely defined linear operators from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. Note that, a simple application of Lemma 5.1 to the operators $\{\mathcal{G}_n\}_{n=1}^\infty$ does not guarantee this point but only that $\mathcal{G} \in \mathcal{L}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$.*

ii) Theorem 5.1 indicates that $\mathcal{L}_{pd}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$ is a closed linear subspace of the Banach space $\mathcal{L}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$.

Proof of Theorem 5.1: We divide the proof into several steps.

Step 1. Since $\{\mathcal{G}_n\}_{n=1}^\infty$ is a sequence of uniformly bounded, pointwisely defined linear operators from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$, for each $n \in \mathbb{N}$ and a.e. $(t, \omega) \in (0, T) \times \Omega$, there exists an $G_n(t, \omega) \in \mathcal{L}(X, Y)$ verifying that

$$(\mathcal{G}_n u(\cdot))(t, \omega) = G_n(t, \omega) u(t, \omega), \quad \forall u(\cdot) \in L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)). \quad (5.3)$$

Write

$$M = \sup_{n \in \mathbb{N}} \|\mathcal{G}_n\|_{\mathcal{L}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))}.$$

By Lemma 5.1, we conclude that there exist a bounded linear operator \mathcal{G} from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$ and a subsequence $\{\mathcal{G}_{n_k}\}_{k=1}^\infty \subset \{\mathcal{G}_n\}_{n=1}^\infty$ such that

$$\mathcal{G}u(\cdot) = (w)\text{-}\lim_{k \rightarrow \infty} \mathcal{G}_{n_k} u(\cdot) \text{ in } L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)), \quad (5.4)$$

and

$$|\mathcal{G}u(\cdot)|_{L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))} \leq M |u(\cdot)|_{L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))}, \quad \forall u(\cdot) \in L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)). \quad (5.5)$$

We claim that

$$\sum_{i=1}^m f_i \mathcal{G}u_i = \mathcal{G}\left(\sum_{i=1}^m f_i u_i\right) = (w)\text{-}\lim_{k \rightarrow \infty} \sum_{i=1}^m f_i \mathcal{G}_{n_k} u_i \text{ in } L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)) \quad (5.6)$$

and

$$\left| \sum_{i=1}^m f_i \mathcal{G}u_i \right|_{L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))} \leq M \left| \sum_{i=1}^m f_i u_i \right|_{L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))}, \quad (5.7)$$

where $m \in \mathbb{N}$, $f_i \in L_{\mathbb{F}}^{\infty}(0, T)$ and $u_i \in L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$, $i = 1, 2, \dots, m$. To show this, write $q'_1 = \frac{q_1}{q_1-1}$ and $q'_2 = \frac{q_2}{q_2-1}$. It follows from (5.4) and (5.3) that for any $v(\cdot) \in L_{\mathbb{F}}^{q'_1}(0, T; L^{q'_1}(\Omega; Y^*))$,

$$\begin{aligned} & \int_0^T \mathbb{E} \left\langle \sum_{i=1}^m f_i(s) (\mathcal{G}u_i)(s), v(s) \right\rangle_{Y, Y^*} ds = \int_0^T \mathbb{E} \sum_{i=1}^m \langle \mathcal{G}u_i(s), f_i(s)v(s) \rangle_{Y, Y^*} ds \\ & = \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \sum_{i=1}^m \langle (\mathcal{G}_{n_k}u_i)(s), f_i(s)v(s) \rangle_{Y, Y^*} ds = \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \sum_{i=1}^m \langle (f_i \mathcal{G}_{n_k}u_i)(s), v(s) \rangle_{Y, Y^*} ds, \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \sum_{i=1}^m \langle \mathcal{G}_{n_k}(s)u_i(s), f_i(s)v(s) \rangle_{Y, Y^*} ds = \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \sum_{i=1}^m \langle G_{n_k}(s)u_i(s), f_i(s)v(s) \rangle_{Y, Y^*} ds \\ & = \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \langle G_{n_k}(s) \left(\sum_{i=1}^m f_i(s)u_i(s) \right), v(s) \rangle_{Y, Y^*} ds \\ & = \lim_{k \rightarrow \infty} \int_0^T \mathbb{E} \langle \left(\mathcal{G}_{n_k} \left(\sum_{i=1}^m f_i u_i \right) \right)(s), v(s) \rangle_{Y, Y^*} ds = \int_0^T \mathbb{E} \langle \left(\mathcal{G} \left(\sum_{i=1}^m f_i u_i \right) \right)(s), v(s) \rangle_{Y, Y^*} ds. \end{aligned} \quad (5.9)$$

By (5.8)–(5.9), we obtain (5.6)–(5.7).

Step 2. Each $x \in X$ can be regarded as an element (i.e., $\chi_{(0, T) \times \Omega}(\cdot)x$) in $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$. Hence, $\mathcal{G}x$ makes sense and belongs to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. It is easy to see that \mathcal{L} is a bounded linear operator from X to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_1}(\Omega; Y))$. By (5.5), we find that

$$|(\mathcal{G}x)(\cdot)|_{L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))} \leq MT^{1/p_2}|x|_X, \quad \forall x \in X. \quad (5.10)$$

Write $B_X = \{x \in X \mid |x|_X \leq 1\}$. By the separability of X , we see that $\left\{ \sup_{x \in B_X} |(\mathcal{G}x)(\cdot)|_Y \right\}$ is an \mathbb{F} -adapted process. We claim that

$$\sup_{x \in B_X} |(\mathcal{G}x)(t, \omega)|_Y < \infty, \quad \text{a.e. } (t, \omega) \in (0, T) \times \Omega. \quad (5.11)$$

In the rest of this step, we shall prove (5.11) by the contradiction argument.

Assume that (5.11) was not true. Then, thanks to the adaptedness of $\left\{ \sup_{x \in B_X} |(\mathcal{G}x)(\cdot)|_Y \right\}$ with respect to \mathbb{F} , there would be a set $A \in \mathcal{M}$, defined by (5.2), such that $\mu(A) > 0$ (Here μ stands for the product measure of the Lebesgue measure (on $[0, T]$) and the probability measure \mathbb{P}) and that

$$\sup_{x \in B_X} |(\mathcal{G}x)(t, \omega)|_Y = \infty, \quad \text{for } (t, \omega) \in A.$$

Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in B_X such that it is dense in B_X . Then

$$\sup_{i \in \mathbb{N}} |(\mathcal{G}x_i)(t, \omega)|_Y = \sup_{x \in B_X} |(\mathcal{G}x)(t, \omega)|_Y = \infty, \quad \text{for } (t, \omega) \in A.$$

For any $n \in \mathbb{N}$, we define a sequence of subsets of $(0, T) \times \Omega$ in the following way.

$$\begin{cases} A_1^{(n)} = \left\{ (t, \omega) \in (0, T) \times \Omega \mid |(\mathcal{G}x_1)(t, \omega)|_Y \geq n \right\}, \\ A_i^{(n)} = \left\{ (t, \omega) \in ((0, T) \times \Omega) \setminus \left(\bigcup_{k=1}^{i-1} A_k^{(n)} \right) \mid |(\mathcal{G}x_i)(t, \omega)|_Y \geq n \right\}, \quad \text{if } i > 1. \end{cases} \quad (5.12)$$

It follows from the adaptedness of $|(\mathcal{G}x)(\cdot)|_Y$ that $A_i^{(n)} \in \mathcal{M}$ for every $i \in \mathbb{N}$ and $n \in \mathbb{N}$. It is clear that $A \subset \bigcup_{i=1}^{\infty} A_i^{(n)}$ for any $n \in \mathbb{N}$ and $A_i^{(n)} \cap A_j^{(n)} = \emptyset$ for $i \neq j$. Hence, we see that

$$\sum_{i=1}^{\infty} \mu(A_i^{(n)}) = \mu\left(\bigcup_{i=1}^{\infty} A_i^{(n)}\right) \geq \mu(A) > 0, \text{ for all } n \in \mathbb{N}.$$

Thus, for each $n \in \mathbb{N}$, there is a $N_n \in \mathbb{N}$ such that

$$\sum_{i=1}^{N_n} \mu(A_i^{(n)}) = \mu\left(\bigcup_{i=1}^{N_n} A_i^{(n)}\right) \geq \frac{\mu(A)}{2} > 0. \quad (5.13)$$

Write

$$x^{(n)}(t, \omega) = \sum_{i=1}^{N_n} \chi_{A_i^{(n)}}(t, \omega) x_i. \quad (5.14)$$

Clearly, $\{x^{(n)}(t)\}_{t \in [0, T]}$ is an adapted process. By $\mathcal{G} \in \mathcal{L}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$ and $|x^{(n)}(t, \omega)|_X \leq 1$ for a.e. $(t, \omega) \in (0, T) \times \Omega$, we find that

$$\begin{aligned} |\mathcal{G}x^{(n)}|_{L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))} &\leq M \left\{ \int_0^T \left[\int_{\Omega} |x^{(n)}(t, \omega)|_X^{p_1} \mathbb{P}(d\omega) \right]^{\frac{p_2}{p_1}} dt \right\}^{\frac{1}{p_2}} \\ &\leq MT^{1/p_2}, \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (5.15)$$

On the other hand, let us choose a $n > \frac{2M}{\mu(A)} T^{\frac{1}{p_1} + \frac{1}{q_1}}$. From (5.12)–(5.14), and noting (5.6), we obtain that

$$\begin{aligned} |\mathcal{G}x^{(n)}|_{L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))} &\geq T^{-\frac{1}{q_1}} |\mathcal{G}x^{(n)}|_{L_{\mathbb{F}}^1(0, T; L^1(\Omega; Y))} \\ &= T^{-\frac{1}{q_1}} \sum_{i=1}^{N_n} \int_{A_i^{(n)}} |\mathcal{G}x_i|_Y dt d\mathbb{P} = T^{-\frac{1}{q_1}} \sum_{i=1}^{N_n} \int_{A_i^{(n)}} |\mathcal{G}x_i|_Y d\mu \\ &\geq T^{-\frac{1}{q_1}} n \sum_{i=1}^{N_n} \mu(A_i^{(n)}) \geq \frac{\mu(A)}{2} T^{-\frac{1}{q_1}} n > MT^{\frac{1}{p_1}}, \end{aligned}$$

which contradicts the inequality (5.15). Therefore, we conclude that (5.11) holds.

Step 3. By (5.11), for a.e. $(t, \omega) \in (0, T) \times \Omega$, we may define an operator $G(t, \omega) \in \mathcal{L}(X, Y)$ by

$$X \ni x \mapsto G(t, \omega)x = (\mathcal{G}x)(t, \omega). \quad (5.16)$$

Further, we introduce the following subspace of $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$:

$$\mathcal{X} = \left\{ u(\cdot) = \sum_{i=1}^m \chi_{A_i}(\cdot) h_i \mid m \in \mathbb{N}, A_i \in \mathcal{M}, h_i \in X \right\}.$$

It is clear that \mathcal{X} is dense in $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$. We now define a linear operator $\tilde{\mathcal{G}}$ from \mathcal{X} to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$ by

$$\mathcal{X} \ni u(\cdot) = \sum_{i=1}^m \chi_{A_i}(\cdot) h_i \mapsto (\tilde{\mathcal{G}}u)(t, \omega) = \sum_{i=1}^m \chi_{A_i}(t, \omega) G(t, \omega) h_i. \quad (5.17)$$

We claim that

$$(\tilde{\mathcal{G}}u)(\cdot) = (\mathcal{G}u)(\cdot), \quad \forall u(\cdot) \in \mathcal{X}. \quad (5.18)$$

Indeed, it follows from (5.3) that for any $v(\cdot) \in L_{\mathbb{F}}^{q'_1}(0, T; L^{q'_2}(\Omega; Y^*))$, and $u(\cdot)$ to be of the form in (5.17),

$$\begin{aligned} \mathbb{E} \int_0^T \langle (\tilde{\mathcal{G}}u)(s), v(s) \rangle_{Y, Y^*} ds &= \mathbb{E} \int_0^T \left\langle \sum_{i=1}^m \chi_{A_i}(s) G(s) h_i, v(s) \right\rangle_{Y, Y^*} ds \\ &= \mathbb{E} \int_0^T \left\langle \sum_{i=1}^m \chi_{A_i}(s) (\mathcal{G}h_i)(s), v(s) \right\rangle_{Y, Y^*} ds = \sum_{i=1}^m \mathbb{E} \int_0^T \langle (\mathcal{G}h_i)(s), \chi_{A_i}(s) v(s) \rangle_{Y, Y^*} ds \\ &= \sum_{i=1}^m \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \langle G_{n_k}(s) h_i, \chi_{A_i}(s) v(s) \rangle_{Y, Y^*} ds \\ &= \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \left\langle G_{n_k}(s) \left(\sum_{i=1}^m \chi_{A_i}(s) h_i \right), v(s) \right\rangle_{Y, Y^*} ds = \mathbb{E} \int_0^T \langle (\mathcal{G}u)(s), v(s) \rangle_{Y, Y^*} ds. \end{aligned}$$

This gives (5.18).

Recall that \mathcal{G} is a bounded linear operator from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. Hence, it is also a bounded linear operator from \mathcal{X} to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. By (5.18), we conclude that $\tilde{\mathcal{G}}$ is a bounded linear operator from \mathcal{X} to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. Since \mathcal{X} is dense in $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$, it is clear that $\tilde{\mathcal{G}}$ can be uniquely extended as a bounded linear operator from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$ (We still denote by $\tilde{\mathcal{G}}$ its extension). By (5.18) again, we conclude that

$$\tilde{\mathcal{G}} = \mathcal{G}. \quad (5.19)$$

It remains to show that

$$(\tilde{\mathcal{G}}u(\cdot))(t, \omega) = G(t, \omega)u(t, \omega), \quad \text{a.e. } (t, \omega) \in (0, T) \times \Omega, \quad (5.20)$$

for all $u \in L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$. For this purpose, by the fact that \mathcal{X} is dense in $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$, we may assume that

$$u(\cdot) = \sum_{i=1}^{\infty} \chi_{A_i}(\cdot) h_i, \quad (5.21)$$

for some $A_i \in \mathcal{M}$ and $h_i \in X$, $i = 1, 2, \dots$ (Note that here we assume neither $A_i \cap A_j = \emptyset$ nor $h_i \neq h_j$ for $i, j = 1, 2, \dots$). For each $n \in \mathbb{N}$, write $u^n(\cdot) = \sum_{i=1}^n \chi_{A_i}(\cdot) h_i$. From (5.21), it is clear that

$$u(\cdot) = \lim_{n \rightarrow \infty} u^n(\cdot), \quad \text{in } L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)). \quad (5.22)$$

By (5.7), (5.16), (5.17) and (5.22), it is easy to see that

$$(\tilde{\mathcal{G}}u^n(\cdot))(t, \omega) = \sum_{i=1}^n \chi_{A_i}(t, \omega) G(t, \omega) h_i \quad (5.23)$$

is a Cauchy sequence in $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. Hence, by (5.23) and recalling that $\tilde{\mathcal{G}}$ is a bounded linear operator from $L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X))$ to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$, we conclude that

$$(\tilde{\mathcal{G}}u(\cdot))(t, \omega) = \sum_{i=1}^{\infty} \chi_{A_i}(t, \omega) G(t, \omega) h_i. \quad (5.24)$$

Combining (5.21) and (5.24), we obtain (5.20).

Finally, by (5.19) and (5.20), the desired result follows. This completes the proof of Theorem 5.1. \square

From the proof of Theorem 5.1, it is easy to deduce the following result.

Corollary 5.1 *Let X and Y be respectively a separable and a reflexive Banach space, and let $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, with $1 \leq p < \infty$, be separable. Let $1 < q_1, q_2 < \infty$. Assume that $\{\mathcal{G}_n\}_{n=1}^\infty$ is a sequence of uniformly bounded, pointwisely defined linear operators from X to $L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y))$. Then, there exist a subsequence $\{\mathcal{G}_{n_k}\}_{k=1}^\infty \subset \{\mathcal{G}_n\}_{n=1}^\infty$ and an $\mathcal{G} \in \mathcal{L}_{pd}(X, L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$ such that*

$$\mathcal{G}x = (w)\text{-}\lim_{k \rightarrow \infty} \mathcal{G}_{n_k}x \quad \text{in } L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)), \quad \forall x \in X.$$

Moreover, $\|\mathcal{G}\|_{\mathcal{L}(X, L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))} \leq \sup_{n \in \mathbb{N}} \|\mathcal{G}_n\|_{\mathcal{L}(X, L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))}$.

Proceeding exactly as in the proof of Theorem 5.1, we can show the following “random variable” and “random variable-stochastic process” versions of Lemma 5.1 (Hence, the detailed proof will be omitted).

Theorem 5.2 *Let X and Y be accordingly a separable and a reflexive Banach space, and let $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, with $1 \leq p < \infty$, be separable. Let $1 \leq p_1 < \infty$ and $1 < q_1 < \infty$. Assume that $\{\mathcal{G}_n\}_{n=1}^\infty$ is a sequence of uniformly bounded, pointwisely defined linear operators from $L_{\mathcal{F}_T}^{p_1}(\Omega; X)$ to $L_{\mathcal{F}_T}^{q_1}(\Omega; Y)$. Then, there exist a subsequence $\{\mathcal{G}_{n_k}\}_{k=1}^\infty \subset \{\mathcal{G}_n\}_{n=1}^\infty$ and an $\mathcal{G} \in \mathcal{L}_{pd}(L_{\mathcal{F}_T}^{p_1}(\Omega; X), L_{\mathcal{F}_T}^{q_1}(\Omega; Y))$ such that*

$$\mathcal{G}u(\cdot) = (w)\text{-}\lim_{k \rightarrow \infty} \mathcal{G}_{n_k}u(\cdot) \quad \text{in } L_{\mathcal{F}_T}^{q_1}(\Omega; Y), \quad \forall u(\cdot) \in L_{\mathcal{F}_T}^{p_1}(\Omega; X).$$

Moreover, $\|\mathcal{G}\|_{\mathcal{L}(L_{\mathcal{F}_T}^{p_1}(\Omega; X), L_{\mathcal{F}_T}^{q_1}(\Omega; Y))} \leq \sup_{n \in \mathbb{N}} \|\mathcal{G}_n\|_{\mathcal{L}(L_{\mathcal{F}_T}^{p_1}(\Omega; X), L_{\mathcal{F}_T}^{q_1}(\Omega; Y))}$.

Theorem 5.3 *Let X and Y be respectively a separable and a reflexive Banach space, and let $L^p(\Omega, \mathcal{F}_T, \mathbb{P})$, with $1 \leq p < \infty$, be separable. Let $1 \leq p_1 < \infty$, $1 < q_1, q_2 < \infty$ and $0 \leq t_0 \leq T$. Assume that $\{\mathcal{G}_n\}_{n=1}^\infty$ is a sequence of uniformly bounded, pointwisely defined linear operators from $L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X)$ to $L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y))$. Then, there exist a subsequence $\{\mathcal{G}_{n_k}\}_{k=1}^\infty \subset \{\mathcal{G}_n\}_{n=1}^\infty$ and an $\mathcal{G} \in \mathcal{L}_{pd}(L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X), L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y)))$ such that*

$$\mathcal{G}u(\cdot) = (w)\text{-}\lim_{k \rightarrow \infty} \mathcal{G}_{n_k}u(\cdot) \quad \text{in } L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y)), \quad \forall u(\cdot) \in L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X).$$

Moreover, $\|\mathcal{G}\|_{\mathcal{L}(L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X), L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y)))} \leq \sup_{n \in \mathbb{N}} \|\mathcal{G}_n\|_{\mathcal{L}(L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X), L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y)))}$.

Remark 5.3 *Similar to Remark 5.2 ii), we see that $\mathcal{L}_{pd}(X, L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$, $\mathcal{L}_{pd}(L_{\mathcal{F}_T}^{p_1}(\Omega; X), L_{\mathcal{F}_T}^{q_1}(\Omega; Y))$ and $\mathcal{L}_{pd}(L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X), L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y)))$ (for any given $t_0 \in [0, T]$) are accordingly closed linear subspaces of the Banach spaces $\mathcal{L}(X, L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$, $\mathcal{L}(L_{\mathcal{F}_T}^{p_1}(\Omega; X), L_{\mathcal{F}_T}^{q_1}(\Omega; Y))$ and $\mathcal{L}(L_{\mathcal{F}_{t_0}}^{p_1}(\Omega; X), L_{\mathbb{F}}^{q_1}(t_0, T; L^{q_2}(\Omega; Y)))$.*

It is clear that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ plays no special role in the above Theorems 5.1–5.3. For possible applications in other places, we give below a “deterministic” modification of Theorem 5.1.

Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two finite measure spaces. Let \mathcal{M} be a sub- σ -field of the σ -field generated by $\mathcal{M}_1 \times \mathcal{M}_2$, and for any $1 \leq p, q < \infty$, let

$$L_{\mathcal{M}}^p(\Omega_1; L^q(\Omega_2; X)) = \left\{ \varphi : \Omega_1 \times \Omega_2 \rightarrow X \mid \varphi(\cdot) \text{ is } \mathcal{M}\text{-measurable and} \right. \\ \left. \int_{\Omega_1} \left(\int_{\Omega_2} |\varphi(\omega_1, \omega_2)|_H^q d\mu_2(\omega_2) \right)^{\frac{p}{q}} d\mu_1(\omega_1) < \infty \right\}.$$

It is easy to show that $L_{\mathcal{M}}^p(\Omega_1; L^q(\Omega_2; X))$ is a Banach space with the canonical norm. Similar to the proof of Theorem 5.1, one can prove the following result:

Theorem 5.4 *Let X and Y be respectively a separable and a reflexive Banach space. Let $1 \leq p_1, p_2 < \infty$ and $1 < q_1, q_2 < \infty$, and let $L_{\mathcal{M}}^{p_1}(\Omega_1; L^{p_2}(\Omega_2; \mathbb{C}))$ be separable. Assume that $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a sequence of uniformly bounded, pointwisely defined linear operators from $L_{\mathcal{M}}^{p_1}(\Omega_1; L^{p_2}(\Omega_2; X))$ to $L_{\mathcal{M}}^{q_1}(\Omega_1; L^{q_2}(\Omega_2; Y))$. Then, there exist a subsequence $\{\mathcal{G}_{n_k}\}_{k=1}^{\infty} \subset \{\mathcal{G}_n\}_{n=1}^{\infty}$ and a $\mathcal{G} \in \mathcal{L}_{pd}(L_{\mathcal{M}}^{p_1}(\Omega_1; L^{p_2}(\Omega_2; X)), L_{\mathcal{M}}^{q_1}(\Omega_1; L^{q_2}(\Omega_2; Y)))$ (defined similarly as $\mathcal{L}_{pd}(L_{\mathbb{F}}^{p_1}(0, T; L^{p_2}(\Omega; X)), L_{\mathbb{F}}^{q_1}(0, T; L^{q_2}(\Omega; Y)))$) such that*

$$\mathcal{G}u(\cdot) = (w)\text{-}\lim_{k \rightarrow \infty} \mathcal{G}_{n_k}u(\cdot) \text{ in } L_{\mathcal{M}}^{q_1}(\Omega_1; L^{q_2}(\Omega_2; Y)), \quad \forall u(\cdot) \in L_{\mathcal{M}}^{p_1}(\Omega_1; L^{p_2}(\Omega_2; X)).$$

Moreover, $\|\mathcal{G}\|_{\mathcal{L}(L_{\mathcal{M}}^{p_1}(\Omega_1; L^{p_2}(\Omega_2; X)), L_{\mathcal{M}}^{q_1}(\Omega_1; L^{q_2}(\Omega_2; Y)))} \leq \sup_{n \in \mathbb{N}} \|\mathcal{G}_n\|_{\mathcal{L}(L_{\mathcal{M}}^{p_1}(\Omega_1; L^{p_2}(\Omega_2; X)), L_{\mathcal{M}}^{q_1}(\Omega_1; L^{q_2}(\Omega_2; Y)))}$.

6 Well-posedness of the operator-valued BSEEs in the general case

This section is addressed to proving the well-posedness result for the equation (1.10) with general data in the sense of relaxed transposition solution, to be defined later.

Write

$$\mathcal{Q}[0, T] \triangleq \left\{ (Q^{(\cdot)}, \widehat{Q}^{(\cdot)}) \mid \text{For any } t \in [0, T], \text{ both } Q^{(t)} \text{ and } \widehat{Q}^{(t)} \text{ are bounded linear operators} \right. \\ \left. \text{from } L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \text{ to } L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)) \right. \\ \left. \text{and } Q^{(t)}(0, 0, \cdot)^* = \widehat{Q}^{(t)}(0, 0, \cdot) \right\}. \quad (6.1)$$

We now define the relaxed transposition solution to (1.10) as follows:

Definition 6.1 *We call $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}) \in D_{\mathbb{F}, w}([0, T]; L^{\frac{4}{3}}(\Omega; \mathcal{L}(H))) \times \mathcal{Q}[0, T]$ a relaxed transposition solution to (1.10) if for any $t \in [0, T]$, $\xi_1, \xi_2 \in L_{\mathcal{F}_t}^4(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$, it holds that*

$$\begin{aligned} & \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \\ &= \mathbb{E} \langle P(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s) u_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) x_1(s), u_2(s) \rangle_H ds \\ &+ \mathbb{E} \int_t^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) v_1(s), K(s) x_2(s) + v_2(s) \rangle_H ds \\ &+ \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds. \end{aligned} \quad (6.2)$$

Here, $x_1(\cdot)$ and $x_2(\cdot)$ solve (1.13) and (1.14), respectively.

Remark 6.1 *It is easy to see that, if $(P(\cdot), Q(\cdot))$ is a transposition solution to (1.10), then $(P(\cdot), Q(\cdot), \widehat{Q}(\cdot))$ is a relaxed transposition solution to the same equation, where (Recall Lemma 2.6 for $U(\cdot, t)$, $V(\cdot, t)$ and $W(\cdot, t)$)*

$$\begin{cases} Q^{(t)}(\xi, u, v) = Q(\cdot)U(\cdot, t)\xi + Q(\cdot)V(\cdot, t)u + \frac{Q(\cdot)W(\cdot, t) + (Q(\cdot)^*W(\cdot, t))^*}{2}v, \\ \widehat{Q}^{(t)}(\xi, u, v) = Q(\cdot)^*U(\cdot, t)\xi + Q(\cdot)^*V(\cdot, t)u + \frac{Q(\cdot)^*W(\cdot, t) + (Q(\cdot)W(\cdot, t))^*}{2}v, \end{cases}$$

for any $(\xi, u, v) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$. However, it is unclear how to obtain a transposition solution $(P(\cdot), Q(\cdot))$ to (1.10) by means of its relaxed transposition solution $(P(\cdot), Q(\cdot), \widehat{Q}(\cdot))$. It seems that this is possible but we cannot do it at this moment.

We have the following well-posedness result for the equation (1.10).

Theorem 6.1 *Assume that H is a separable Hilbert space, and $L_{\mathcal{F}_T}^p(\Omega; \mathbb{C})$ ($1 \leq p < \infty$) is a separable Banach space. Then, for any $P_T \in L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))$, $F \in L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))$ and $J, K \in L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, the equation (1.10) admits one and only one relaxed transposition solution $(P(\cdot), Q(\cdot), \widehat{Q}(\cdot)) \in D_{\mathbb{F}, w}([0, T]; L^{\frac{4}{3}}(\Omega; \mathcal{L}(H))) \times \mathcal{Q}[0, T]$. Furthermore,*

$$\begin{aligned} & \|P\|_{\mathcal{L}(L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & + \sup_{t \in [0, T]} \|(Q^{(t)}, \widehat{Q}^{(t)})\|_{(\mathcal{L}(L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)))}^2 \\ & \leq C \left[|F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))} + |P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} \right]. \end{aligned} \quad (6.3)$$

Proof: We consider only the case that H is a real Hilbert space (The case of complex Hilbert spaces can be treated similarly). The proof is divided into several steps.

Step 1. In this step, we introduce a suitable approximation to the equation (1.10).

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of H and $\{\Gamma_n\}_{n=1}^\infty$ be the standard projection operator from H onto its subspace $\text{span}\{e_1, e_2, \dots, e_n\}$, that is, $\Gamma_n x = \sum_{i=1}^n x_i e_i$ for any $x = \sum_{i=1}^\infty x_i e_i \in H$. Write $H_n = \Gamma_n H$. It is clear that, for each $n \in \mathbb{N}$, H_n is isomorphic to the n -dimensional Euclidean space \mathbb{R}^n . In the sequel, we identify H_n by \mathbb{R}^n , and hence $\mathcal{L}(H_n) = \mathcal{L}(\mathbb{R}^n)$ is the set of all $n \times n$ (real) matrices. For any $M_1, M_2 \in \mathcal{L}(\mathbb{R}^n)$, put $\langle M_1, M_2 \rangle_{\mathcal{L}(\mathbb{R}^n)} = \text{tr}(M_1 M_2^\top)$. It is easy to check that $\langle \cdot, \cdot \rangle_{\mathcal{L}(\mathbb{R}^n)}$ is an inner product on $\mathcal{L}(\mathbb{R}^n)$, and $\mathcal{L}(\mathbb{R}^n)$ is a Hilbert space with this inner product.

Consider the following matrix-valued BSDE:

$$\begin{cases} dP^{n, \lambda} = -(A_{\lambda, n}^* + J_n^*)P^{n, \lambda}dt - P^{n, \lambda}(A_{\lambda, n} + J_n)dt - K_n^*P^{n, \lambda}K_ndt \\ \quad - (K_n^*Q^{n, \lambda} + Q^{n, \lambda}K_n)dt + F_n dt + Q^{n, \lambda}dw(t) \\ P^{n, \lambda}(T) = P_T^n, \end{cases} \quad \text{in } [0, T], \quad (6.4)$$

where $\lambda \in \rho(A)$, $A_{\lambda, n} = \Gamma_n A_\lambda \Gamma_n$, A_λ (as before) stands for the Yosida approximation of A , $J_n = \Gamma_n J \Gamma_n$, $K_n = \Gamma_n K \Gamma_n$, $F_n = \Gamma_n F \Gamma_n$ and $P_T^n = \Gamma_n P_T \Gamma_n$.

The solution to (6.4) is understood in the transposition sense. According to Theorem 3.1 (or [18, Theorem 4.1]), the equation (6.4) admits a unique transposition solution $(P^{n, \lambda}(\cdot), Q^{n, \lambda}(\cdot)) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}(\mathbb{R}^n))) \times L_{\mathbb{F}}^2(0, T; L^2(\Omega; \mathcal{L}(\mathbb{R}^n)))$ such that, for any $t \in [0, T]$, $U_1^n(\cdot) \in L_{\mathbb{F}}^1(t, T;$

$L^2(\Omega; \mathcal{L}(\mathbb{R}^n))$, $V_1^n(\cdot) \in L^2_{\mathbb{F}}(t, T; L^2(\Omega; \mathcal{L}(\mathbb{R}^n)))$ and $\eta \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{L}(\mathbb{R}^n))$, and the corresponding solution $X^n(\cdot) \in C_{\mathbb{F}}([t, T]; L^2(\Omega; \mathcal{L}(\mathbb{R}^n)))$ of the following equation:

$$\begin{cases} dX^n = U_1^n ds + V_1^n dw(s) & \text{in } (t, T], \\ X^n(t) = \eta, \end{cases} \quad (6.5)$$

it holds that

$$\begin{aligned} & \mathbb{E} \langle X^n(T), P_T^n \rangle_{\mathcal{L}(\mathbb{R}^n)} - \mathbb{E} \int_t^T \langle X^n(s), \Phi^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds \\ &= \mathbb{E} \langle \eta, P^{n,\lambda}(t) \rangle_{\mathcal{L}(\mathbb{R}^n)} + \mathbb{E} \int_t^T \langle U_1^n(s), P^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds + \mathbb{E} \int_t^T \langle V_1^n(s), Q^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds, \end{aligned} \quad (6.6)$$

where

$$\Phi^{n,\lambda} = -(A_{\lambda,n}^* + J_n^*)P^{n,\lambda} - P^{n,\lambda}(A_{\lambda,n} + J_n) - K_n^*P^{n,\lambda}K_n - K_n^*Q^{n,\lambda} - Q^{n,\lambda}K_n + F_n. \quad (6.7)$$

Clearly, (6.4) can be regarded as finite dimensional approximations of the equation (1.10). In the rest of the proof, we shall construct the desired solution to the equation (1.10) by means of the solutions to (6.4).

Step 2. This step is devoted to introducing suitable finite approximations of the equations (1.13) and (1.14).

We approximate accordingly (1.13) and (1.14) by the following finite dimensional systems:

$$\begin{cases} dx_1^{n,\lambda} = (A_{\lambda,n} + J_n)x_1^{n,\lambda} ds + u_1^n ds + K_n x_1^{n,\lambda} dw(s) + v_1^n dw(s) & \text{in } (t, T], \\ x_1^{n,\lambda}(t) = \xi_1^n \end{cases} \quad (6.8)$$

and

$$\begin{cases} dx_2^{n,\lambda} = (A_{\lambda,n} + J_n)x_2^{n,\lambda} ds + u_2^n ds + K_n x_2^{n,\lambda} dw(s) + v_2^n dw(s) & \text{in } (t, T], \\ x_2^{n,\lambda}(t) = \xi_2^n. \end{cases} \quad (6.9)$$

Here $\xi_1^n = \Gamma_n \xi_1$, $\xi_2^n = \Gamma_n \xi_2$, $u_1^n(\cdot) = \Gamma_n u_1(\cdot)$, $u_2^n(\cdot) = \Gamma_n u_2(\cdot)$, $v_1^n(\cdot) = \Gamma_n v_1(\cdot)$ and $v_2^n(\cdot) = \Gamma_n v_2(\cdot)$. It is easy to see that both (6.8) and (6.9) are stochastic differential equations. Obviously, $\xi_1^n, \xi_2^n \in L^4_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$, $u_1^n, u_2^n \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; \mathbb{R}^n))$ and $v_1^n, v_2^n \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; \mathbb{R}^n))$. One can easily check that, for $k = 1, 2$,

$$\begin{cases} \lim_{n \rightarrow \infty} \xi_k^n = \xi_k \text{ in } L^4_{\mathcal{F}_t}(\Omega; H), \\ \lim_{n \rightarrow \infty} u_k^n = u_k \text{ in } L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)), \\ \lim_{n \rightarrow \infty} v_k^n = v_k \text{ in } L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)). \end{cases} \quad (6.10)$$

Then, similar to Lemma 2.7, one can show that

$$\lim_{n \rightarrow \infty} x_k^{n,\lambda} = x_k^\lambda \text{ in } L^4_{\mathbb{F}}(\Omega; C([t, T]; H)), \quad k = 1, 2. \quad (6.11)$$

Hence, by Lemma 2.7, we obtain that

$$\lim_{\lambda \rightarrow \infty} \lim_{n \rightarrow \infty} x_k^{n,\lambda} = x_k \text{ in } C_{\mathbb{F}}([t, T]; L^4(\Omega; H)), \quad k = 1, 2. \quad (6.12)$$

Denote by $U_{n,\lambda}(\cdot, \cdot)$ the bounded linear operator such that $U_{n,\lambda}(\cdot, t)\xi_1^n$ solves the equation (6.8) with $u_1^n = v_1^n = 0$. Clearly, $U_{n,\lambda}(\cdot, t)\xi_2^n$ solves the equation (6.9) with $u_2^n = v_2^n = 0$. We claim that that for any $\lambda \in \rho(A)$, there is a constant $C(\lambda) > 0$ such that for all $n \in \mathbb{N}$, it holds that

$$|U_{n,\lambda}(\cdot, t)\xi_1^n|_{L^\infty_{\mathbb{F}}(t, T; L^4(\Omega; H))} \leq C(\lambda)|\xi_1|_{L^4_{\mathcal{F}_t}(\Omega; H)}. \quad (6.13)$$

Indeed, by

$$x_1^{n,\lambda}(s) = S_{n,\lambda}(s-t)\xi_1^n + \int_t^s S_{n,\lambda}(s-\tau)J_n(\tau)x_1^{n,\lambda}(\tau)d\tau + \int_t^s S_{n,\lambda}(s-\tau)K_n(\tau)x_1^{n,\lambda}(\tau)dw(\tau),$$

noting that for all $n \in \mathbb{N}$, $|S_{n,\lambda}(\cdot)|_{L^\infty(0,T;\mathcal{L}(H))} \leq e^{\|A_\lambda\|_{\mathcal{L}(H)}T}$ and utilizing Lemma 2.1, we obtain that

$$\begin{aligned} & \mathbb{E}|U_{n,\lambda}(s,t)\xi_1^n|_H^4 \\ &= \mathbb{E}\left|S_{n,\lambda}(s-t)\xi_1^n + \int_t^s S_{n,\lambda}(s-\tau)J(\tau)U_{n,\lambda}(\tau,t)\xi_1^n d\tau + \int_t^s S_{n,\lambda}(s-\tau)K(\tau)U_{n,\lambda}(\tau,t)\xi_1^n dw\right|_H^4 \\ &\leq C(\lambda)\mathbb{E}|\xi_1|_H^4 + C(\lambda)\int_t^s \left[|J(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4\right]|U_{n,\lambda}(\tau,t)\xi_1^n|_H^4 d\tau. \end{aligned}$$

This, together with Gronwall's inequality, implies (6.13).

Also, denote by $U_\lambda(\cdot, \cdot)$ the bounded linear operator such that $U_\lambda(\cdot, t)\xi_1$ solves the equation (2.11) with $u_1 = v_1 = 0$. Clearly, $U_\lambda(\cdot, t)\xi_2$ solves the equation (2.12) with $u_2 = v_2 = 0$. We claim that there is a constant $C > 0$ such that for any $\lambda \in \rho(A)$ it holds that

$$|U_\lambda(\cdot, t)\xi_1|_{L_{\mathbb{F}}^\infty(t,T;L^4(\Omega;H))} \leq C|\xi_1|_{L_{\mathbb{F}_t}^4(\Omega;H)}. \quad (6.14)$$

Indeed, similar to the above proof of (6.13), by (2.14) and utilizing Lemma 2.1, we obtain that

$$\begin{aligned} & \mathbb{E}|U_\lambda(s,t)\xi_1|_H^4 \\ &= \mathbb{E}\left|S_\lambda(s-t)\xi_1 + \int_t^s S_\lambda(s-\tau)J(\tau)U_\lambda(\tau,t)\xi_1 d\tau + \int_t^s S_\lambda(s-\tau)K(\tau)U_\lambda(\tau,t)\xi_1 dw\right|_H^4 \\ &\leq C\mathbb{E}|\xi_1|_H^4 + C\int_t^s \left[|J(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4\right]|U_\lambda(\tau,t)\xi_1|_H^4 d\tau. \end{aligned}$$

Hence (6.14) follows from Gronwall's inequality.

Step 3. In this step, we show that $(P^{n,\lambda}(\cdot), Q^{n,\lambda}(\cdot))$ satisfies a variational equality, which can be viewed as an approximation of (1.17).

Denote by $X^{n,\lambda}$ the tensor product of $x_1^{n,\lambda}$ and $x_2^{n,\lambda}$, i.e., $X^{n,\lambda} = x_1^{n,\lambda} \otimes x_2^{n,\lambda}$. Since

$$\begin{aligned} & d(x_1^{n,\lambda} \otimes x_2^{n,\lambda}) \\ &= (dx_1^{n,\lambda}) \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes d(x_2^{n,\lambda}) + (dx_1^{n,\lambda}) \otimes d(x_2^{n,\lambda}) \\ &= \left[(A_{\lambda,n} + J_n)x_1^{n,\lambda}\right] \otimes x_2^{n,\lambda} ds + x_1^{n,\lambda} \otimes \left[(A_{\lambda,n} + J_n)x_2^{n,\lambda}\right] ds \\ &\quad + \left[u_1^n \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes u_2^n + (K_n x_1^{n,\lambda}) \otimes (K_n x_2^{n,\lambda}) + (K_n x_1^{n,\lambda}) \otimes v_2^n + v_1^n \otimes (K_n x_2^{n,\lambda}) + v_1^n \otimes v_2^n\right] ds \\ &\quad + \left[K_n x_1^{n,\lambda} \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes (K_n x_2^{n,\lambda}) + v_1^n \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes v_2^n\right] dw(s), \end{aligned}$$

we see that $X^{n,\lambda}$ solves the following equation:

$$\begin{cases} dX^{n,\lambda} = \alpha^{n,\lambda} ds + \beta^{n,\lambda} dw(s) & \text{in } (t, T], \\ X^{n,\lambda}(t) = \xi_1^n \otimes \xi_2^n, \end{cases} \quad (6.15)$$

where

$$\begin{cases} \alpha^{n,\lambda} = \left[(A_{\lambda,n} + J_n)x_1^{n,\lambda}\right] \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes \left[(A_{\lambda,n} + J_n)x_2^{n,\lambda}\right] + u_1^n \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes u_2^n \\ \quad + (K_n x_1^{n,\lambda}) \otimes (K_n x_2^{n,\lambda}) + (K_n x_1^{n,\lambda}) \otimes v_2^n + v_1^n \otimes (K_n x_2^{n,\lambda}) + v_1^n \otimes v_2^n, \\ \beta^{n,\lambda} = K_n x_1^{n,\lambda} \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes (K_n x_2^{n,\lambda}) + v_1^n \otimes x_2^{n,\lambda} + x_1^{n,\lambda} \otimes v_2^n. \end{cases}$$

Recalling that $(P^{n,\lambda}(\cdot), Q^{n,\lambda}(\cdot))$ is the transposition solution to (6.4), by (6.6) and (6.15), we obtain that

$$\begin{aligned}
& \mathbb{E} \langle x_1^{n,\lambda}(T) \otimes x_2^{n,\lambda}(T), P_T^n \rangle_{\mathcal{L}(\mathbb{R}^n)} - \mathbb{E} \int_t^T \langle x_1^{n,\lambda}(s) \otimes x_2^{n,\lambda}(s), \Phi^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds \\
&= \langle \xi_1 \otimes \xi_2, P^{n,\lambda}(t) \rangle_{\mathcal{L}(\mathbb{R}^n)} + \mathbb{E} \int_t^T \langle \alpha^{n,\lambda}(s), P^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds \\
&+ \mathbb{E} \int_t^T \langle \beta^{n,\lambda}(s), Q^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds,
\end{aligned} \tag{6.16}$$

where $\Phi^{n,\lambda}(\cdot)$ is given by (6.7).

A direct computation shows that

$$\begin{aligned}
& \mathbb{E} \int_t^T \langle x_1^{n,\lambda}(s) \otimes x_2^{n,\lambda}(s), \Phi^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds = \mathbb{E} \int_t^T \langle \Phi^{n,\lambda}(s) x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&= -\mathbb{E} \int_t^T \langle P^{n,\lambda} x_1^{n,\lambda}(s), (A_{\lambda,n} + J_n) x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds - \mathbb{E} \int_t^T \langle P^{n,\lambda} (A_{\lambda,n} + J_n) x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&- \mathbb{E} \int_t^T \langle P^{n,\lambda} K_n x_1^{n,\lambda}(s), K_n x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds - \mathbb{E} \int_t^T \langle Q^{n,\lambda} x_1^{n,\lambda}(s), K_n x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&- \mathbb{E} \int_t^T \langle Q^{n,\lambda} K_n x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle F_n x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds.
\end{aligned} \tag{6.17}$$

Next,

$$\begin{aligned}
& \mathbb{E} \int_t^T \langle \alpha^{n,\lambda}(s), P^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds \\
&= \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) (A_{\lambda,n} + J_n) x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) x_1^{n,\lambda}(s), (A_{\lambda,n} + J_n) x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) u_1^n(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) x_1^{n,\lambda}(s), u_2^n(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) K_n x_1^{n,\lambda}(s), K_n x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) K_n x_1^{n,\lambda}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) v_1^n(s), K_n x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) v_1^n(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds.
\end{aligned} \tag{6.18}$$

Further,

$$\begin{aligned}
& \mathbb{E} \int_t^T \langle \beta^{n,\lambda}(s), Q^{n,\lambda}(s) \rangle_{\mathcal{L}(\mathbb{R}^n)} ds \\
&= \mathbb{E} \int_t^T \langle Q^{n,\lambda}(s) K_n x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle Q^{n,\lambda}(s) x_1^{n,\lambda}(s), K_n x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle Q^{n,\lambda}(s) v_1^n(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle Q^{n,\lambda}(s) x_1^{n,\lambda}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds.
\end{aligned} \tag{6.19}$$

From (6.16)–(6.19), we arrive at

$$\begin{aligned}
& \mathbb{E} \langle P_T^n x_1^{n,\lambda}(T), x_2^{n,\lambda}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_t^T \langle F_n(s) x_1^{n,\lambda}(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&= \mathbb{E} \langle P^{n,\lambda}(t) \xi_1^n, \xi_2^n \rangle_{\mathbb{R}^n} + \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) u_1^n(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) x_1^{n,\lambda}(s), u_2^n(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) K_n(s) x_1^{n,\lambda}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle P^{n,\lambda}(s) v_1^{n,\lambda}(s), K_n(s) x_2^{n,\lambda}(s) + v_2^n(s) \rangle_{\mathbb{R}^n} ds \\
&+ \mathbb{E} \int_t^T \langle Q^{n,\lambda}(s) v_1^n(s), x_2^{n,\lambda}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle Q^{n,\lambda}(s) x_1^{n,\lambda}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds.
\end{aligned} \tag{6.20}$$

From the above $\mathbb{R}^{n \times n}$ -valued processes $P^{n,\lambda}(\cdot)$ and $Q^{n,\lambda}(\cdot)$, one obtains two $\mathcal{L}(H)$ -valued processes $P^{n,\lambda}(\cdot)\Gamma_n$ and $Q^{n,\lambda}(\cdot)\Gamma_n$. To simplify the notations, we simply identify $P^{n,\lambda}(\cdot)$ (*resp.* $Q^{n,\lambda}(\cdot)$) and $P^{n,\lambda}(\cdot)\Gamma_n$ (*resp.* $Q^{n,\lambda}(\cdot)\Gamma_n$).

Step 4. In this step, we take $n \rightarrow \infty$ in (6.20) with $t \in \{r_j\}_{j=1}^\infty$. Here $\{r_j\}_{j=1}^\infty$ stands for the subset of all rational numbers in $[0, T]$.

In the sequel, we fix a sequence $\{\lambda_m\}_{m=1}^\infty \subset \rho(A)$ such that $\lambda_m \rightarrow \infty$ as $m \rightarrow \infty$.

Choose $u_1^n = v_1^n = 0$ and $u_2^n = v_2^n = 0$ in (6.8) and (6.9), respectively. From the equality (6.20), it follows that

$$\mathbb{E} \langle P_T^n x_1^{n,\lambda_m}(T), x_2^{n,\lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_t^T \langle F_n(s) x_1^{n,\lambda_m}(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds = \mathbb{E} \langle P^{n,\lambda_m}(t) \xi_1^n, \xi_2^n \rangle_{\mathbb{R}^n}. \tag{6.21}$$

Combing (6.13) and (6.21), we find that

$$\begin{aligned}
& \left| \mathbb{E} \langle P^{n,\lambda_m}(t) \xi_1^n, \xi_2^n \rangle_H \right| = \left| \mathbb{E} \langle P^{n,\lambda_m}(t) \xi_1^n, \xi_2^n \rangle_{\mathbb{R}^n} \right| \\
& \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}) |\xi_1|_{L_{\mathcal{F}_t}^4(\Omega; H)} |\xi_2|_{L_{\mathcal{F}_t}^4(\Omega; H)}.
\end{aligned} \tag{6.22}$$

Here and henceforth $C(\lambda_m)$ denotes a generic constant depending only on λ_m , independent of n .

For $P^{n,\lambda_m}(t)$, we can find a $\xi_{1,n,m} \in L_{\mathcal{F}_t}^4(\Omega; H)$ with $|\xi_{1,n,m}|_{L_{\mathcal{F}_t}^4(\Omega; H)} = 1$ such that

$$|P^{n,\lambda_m}(t) \xi_{1,n,m}|_{L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega; H)} \geq \frac{1}{2} |P^{n,\lambda_m}(t)|_{L_{\mathcal{F}_t}^2(\Omega; \mathcal{L}(H))}. \tag{6.23}$$

Moreover, we can find a $\xi_{2,n,m} \in L_{\mathcal{F}_t}^4(\Omega; H)$ with $|\xi_{2,n,m}|_{L_{\mathcal{F}_t}^4(\Omega; H)} = 1$ such that

$$\mathbb{E} \langle P^{n,\lambda_m}(t) \xi_{1,n,m}, \xi_{2,n,m} \rangle_{\mathbb{R}^n} \geq \frac{1}{2} |P^{n,\lambda_m}(t) \xi_1^{n,m}|_{L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega; H)}. \tag{6.24}$$

From (6.22)–(6.24), we obtain that

$$|P^{n,\lambda_m}|_{L_{\mathbb{F}}^\infty(0, T; L^2(\Omega; \mathcal{L}(H)))} \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}), \quad \forall n \in \mathbb{N}. \tag{6.25}$$

Thanks to Theorem 5.1, one can find a $P^{\lambda_m} \in \mathcal{L}_{pd}(L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)))$ such that

$$\|P^{\lambda_m}\|_{\mathcal{L}(L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)))} \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}), \tag{6.26}$$

and a subsequence $\{n_k^{(1)}\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty$ such that

$$(w)\text{-} \lim_{k \rightarrow \infty} P^{n_k^{(1)}, \lambda_m} u = P^{\lambda_m} u \quad \text{in } L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall u \in L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)). \quad (6.27)$$

Note that, by means of the standard diagonalisation argument, one can choose the subsequence $\{n_k^{(1)}\}_{k=1}^\infty$ to be independent of λ_m .

Next, by Theorem 5.2, for each r_j and λ_m , there exist an $R^{(r_j, \lambda_m)} \in \mathcal{L}_{pd}(L_{\mathcal{F}_{r_j}}^4(\Omega; H), L_{\mathcal{F}_{r_j}}^{\frac{4}{3}}(\Omega; H))$ and a subsequence $\{n_k^{(2)}\}_{k=1}^\infty \subset \{n_k^{(1)}\}_{k=1}^\infty$ such that

$$(w)\text{-} \lim_{k \rightarrow \infty} P^{n_k^{(2)}, \lambda_m}(r_j) \xi = R^{(r_j, \lambda_m)} \xi \quad \text{in } L_{\mathcal{F}_{r_j}}^{\frac{4}{3}}(\Omega; H), \quad \forall \xi \in L_{\mathcal{F}_{r_j}}^4(\Omega; H). \quad (6.28)$$

Here, again, by the diagonalisation argument, one can choose the subsequence $\{n_k^{(2)}\}_{k=1}^\infty$ to be independent of r_j and λ_m .

Let $u_1^n = v_1^n = 0$ and $\xi_2^n = 0$, $u_2^n = 0$ in (6.8) and (6.9), respectively. From (6.20), we find that

$$\begin{aligned} & \mathbb{E} \int_t^T \langle Q^{n, \lambda_m}(s) U_{n, \lambda_m}(s, t) \xi_1^n, v_2^n(s) \rangle_H ds \\ &= \mathbb{E} \int_t^T \langle Q^{n, \lambda_m}(s) U_{n, \lambda_m}(s, t) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \langle P_T^n x_1^{n, \lambda_m}(T), x_2^{n, \lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_t^T \langle F_n(s) x_1^{n, \lambda_m}(s), x_2^{n, \lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ & \quad - \mathbb{E} \int_t^T \langle P^{n, \lambda_m}(s) K_n(s) x_1^{n, \lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned}$$

This implies that

$$\begin{aligned} & \mathbb{E} \int_t^T \langle Q^{n, \lambda_m}(s) U_{n, \lambda_m}(s, t) \xi_1^n, v_2^n(s) \rangle_H ds \\ & \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}) \|\xi_1\|_{L_{\mathcal{F}_t}^4(\Omega; H)} \|v_2\|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))}. \end{aligned} \quad (6.29)$$

We define two operators $Q_1^{n, \lambda_m, t}$ and $\widehat{Q}_1^{n, \lambda_m, t}$ from $L_{\mathcal{F}_t}^4(\Omega; H)$ to $L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$ as follows:

$$\begin{cases} Q_1^{n, \lambda_m, t} \xi = Q^{n, \lambda_m}(\cdot) U_{n, \lambda_m}(\cdot, t) \xi^n, & \forall \xi \in L_{\mathcal{F}_t}^4(\Omega; H); \\ \widehat{Q}_1^{n, \lambda_m, t} \xi = Q^{n, \lambda_m}(\cdot)^* U_{n, \lambda_m}(\cdot, t) \xi^n, & \forall \xi \in L_{\mathcal{F}_t}^4(\Omega; H). \end{cases}$$

Here $\xi^n = \Gamma_n \xi$. It is clear that $Q_1^{n, \lambda_m, t}, \widehat{Q}_1^{n, \lambda_m, t} \in \mathcal{L}(L_{\mathcal{F}_t}^4(\Omega; H), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)))$. For any given n , λ_m and t , we can find a $\xi_1^{n, m, t} \in L_{\mathcal{F}_t}^4(\Omega; H)$ with $\|\xi_1^{n, m, t}\|_{L_{\mathcal{F}_t}^4(\Omega; H)} = 1$, such that

$$\|Q_1^{n, \lambda_m, t} \xi_1^{n, m, t}\|_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))} \geq \frac{1}{2} \|Q_1^{n, \lambda_m, t}\|_{\mathcal{L}(L_{\mathcal{F}_t}^4(\Omega; H), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)))}. \quad (6.30)$$

Furthermore, we can find a $v_2^{n, m, t}(\cdot) \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ with $\|v_2^{n, m, t}(\cdot)\|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))} = 1$ such that

$$\mathbb{E} \int_t^T \langle Q^{n, \lambda_m, t}(s) U_{n, \lambda_m}(s, t) \xi_1^{n, m, t}, v_2^{n, m, t}(s) \rangle_H ds \geq \frac{1}{2} \|Q_1^{n, \lambda_m, t} \xi_1^{n, m, t}\|_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))}. \quad (6.31)$$

Hence, combining (6.29), (6.30) and (6.31), it follows that

$$\|Q_1^{n,\lambda_m,t}\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega;H), L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H)))} \leq C(\lambda_m)(|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))}). \quad (6.32)$$

Similarly,

$$\|\widehat{Q}_1^{n,\lambda_m,t}\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega;H), L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H)))} \leq C(\lambda_m)(|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))}). \quad (6.33)$$

By Lemma 5.1, for each r_j and λ_m , there exist two bounded linear operators $Q_1^{\lambda_m,r_j}$ and $\widehat{Q}_1^{\lambda_m,r_j}$, from $L^4_{\mathcal{F}_{r_j}}(\Omega;H)$ to $L^{\frac{4}{3}}_{\mathbb{F}}(r_j,T;L^{\frac{4}{3}}(\Omega;H))$, and a subsequence $\{n_k^{(3)}\}_{k=1}^\infty \subset \{n_k^{(2)}\}_{n=1}^\infty$, independent of r_j and λ_m , such that

$$\begin{cases} \text{(w)-} \lim_{k \rightarrow \infty} Q_1^{n_k^{(3)},\lambda_m,r_j} \xi = Q_1^{\lambda_m,r_j} \xi & \text{in } L^2_{\mathbb{F}}(r_j,T;L^{\frac{4}{3}}(\Omega;H)), \quad \forall \xi \in L^4_{\mathcal{F}_{r_j}}(\Omega;H), \\ \text{(w)-} \lim_{k \rightarrow \infty} \widehat{Q}_1^{n_k^{(3)},\lambda_m,r_j} \xi = \widehat{Q}_1^{\lambda_m,r_j} \xi & \text{in } L^2_{\mathbb{F}}(r_j,T;L^{\frac{4}{3}}(\Omega;H)), \quad \forall \xi \in L^4_{\mathcal{F}_{r_j}}(\Omega;H). \end{cases} \quad (6.34)$$

Next, we choose $\xi_1^n = 0$, $v_1^n = 0$ in (6.8) and $\xi_2^n = 0$, $u_2^n = 0$ in (6.9). From (6.20), we obtain that

$$\begin{aligned} & \mathbb{E} \langle P_T^n x_1^{n,\lambda_m}(T), x_2^{n,\lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_t^T \langle F_n(s) x_1^{n,\lambda_m}(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \int_t^T \langle P^{n,\lambda_m}(s) u_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle P^{n,\lambda_m}(s) K_n(s) x_1^{n,\lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_t^T \langle Q^{n,\lambda_m}(s) x_1^{n,\lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned} \quad (6.35)$$

Define an operator $Q_2^{n,\lambda_m,t}$ from $L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))$ to $L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H))$ as follows:

$$(Q_2^{n,\lambda_m,t}u)(\cdot) = Q^{n,\lambda_m}(\cdot) \int_t^\cdot U_{n,\lambda_m}(\cdot,\tau) u^n(\tau) d\tau, \quad \forall u \in L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)),$$

where $u^n = \Gamma_n u$. From (6.35), we get that

$$\begin{aligned} & \mathbb{E} \int_t^T \langle (Q_2^{n,\lambda_m,t}u_1^n)(s), v_2^n(s) \rangle_H ds = \int_t^T \langle (Q_2^{n,\lambda_m,t}u_1^n)(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \langle P_T^n x_1^{n,\lambda_m}(T), x_2^{n,\lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_t^T \langle F_n(s) x_1^{n,\lambda_m}(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ & - \mathbb{E} \int_t^T \langle P^{n,\lambda_m}(s) u_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds - \mathbb{E} \int_t^T \langle P^{n,\lambda_m}(s) K_n(s) x_1^{n,\lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ & \leq C(\lambda_m)(|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))}) |u_1|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))} |v_2|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))}. \end{aligned} \quad (6.36)$$

Let us choose a $u_1^{n,m,t} \in L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))$ satisfying $|u_1^{n,m,t}|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))} = 1$, such that

$$|Q_2^{n,\lambda_m,t}u_1^{n,m,t}|_{L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H))} \geq \frac{1}{2} \|Q_2^{n,\lambda_m,t}\|_{\mathcal{L}(L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)), L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H)))}. \quad (6.37)$$

Then we choose a $v_2^{n,m,t} \in L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))$ satisfying $|v_2^{n,m,t}|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))} = 1$, such that

$$\mathbb{E} \int_t^T \langle (Q_2^{n,\lambda_m,t}u_1^{n,m,t})(s), v_2^{n,m,t}(s) \rangle_H ds \geq \frac{1}{2} |Q_2^{n,\lambda_m,t}u_1^{n,m,t}|_{L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H))}. \quad (6.38)$$

From (6.36)–(6.38), we see that

$$\|Q_2^{n,\lambda_m,t}\|_{\mathcal{L}(L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)), L^{\frac{4}{3}}_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H)))} \leq C(\lambda_m)(|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))}). \quad (6.39)$$

Also, we define an operator $\widehat{Q}_2^{n,\lambda_m,t}$ from $L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))$ to $L^2_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H))$ as follows:

$$(\widehat{Q}_2^{n,\lambda_m,t}u)(\cdot) = Q^{n,\lambda_m}(\cdot)^* \int_t^\cdot U_{n,\lambda_m}(\cdot, \tau) u^n(\tau) d\tau, \quad \forall u \in L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)),$$

where $u^n = \Gamma_n u$. By a similar argument to derive the inequality (6.39), we find that

$$\|\widehat{Q}_2^{n,\lambda_m,t}\|_{\mathcal{L}(L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)), L^{\frac{4}{3}}_{\mathbb{F}}(t,T;L^{\frac{4}{3}}(\Omega;H)))} \leq C(\lambda_m)(|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))}). \quad (6.40)$$

By Lemma 5.1, we conclude that, for each r_j and λ_m , there exist two bounded linear operators $Q_2^{\lambda_m,r_j}$ and $\widehat{Q}_2^{\lambda_m,r_j}$ from $L^2_{\mathbb{F}}(r_j,T;L^4(\Omega;H))$ to $L^2_{\mathbb{F}}(r_j,T;L^{\frac{4}{3}}(\Omega;H))$ and a subsequence $\{n_k^{(4)}\}_{k=1}^\infty \subset \{n_k^{(3)}\}_{n=1}^\infty$, independent of r_j and λ_m , such that

$$\begin{cases} (w)\text{-}\lim_{k \rightarrow \infty} Q_2^{n_k^{(4)},\lambda_m,r_j} u = Q_2^{\lambda_m,r_j} u & \text{in } L^2_{\mathbb{F}}(r_j,T;L^{\frac{4}{3}}(\Omega;H)), \quad \forall u \in L^2_{\mathbb{F}}(r_j,T;L^4(\Omega;H)), \\ (w)\text{-}\lim_{k \rightarrow \infty} \widehat{Q}_2^{n_k^{(4)},\lambda_m,r_j} u = \widehat{Q}_2^{\lambda_m,r_j} u & \text{in } L^2_{\mathbb{F}}(r_j,T;L^{\frac{4}{3}}(\Omega;H)), \quad \forall u \in L^2_{\mathbb{F}}(r_j,T;L^4(\Omega;H)). \end{cases} \quad (6.41)$$

Now, we choose $\xi_1^n = 0$ and $u_1^n = 0$ in (6.8), and $\xi_2^n = 0$ and $u_2^n = 0$ in (6.9). From (6.20), we obtain that

$$\begin{aligned} & \mathbb{E} \langle P_T^n x_1^{n,\lambda_m}(T), x_2^{n,\lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_t^T \langle F_n(s) x_1^{n,\lambda_m}(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \int_t^T \langle P^{n,\lambda_m}(s) K_n(s) x_1^{n,\lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle P^{n,\lambda_m}(s) v_1^n(s), K_n(s) x_2^{n,\lambda_m}(s) + v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_t^T \langle Q^{n,\lambda_m}(s) v_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle Q^{n,\lambda_m}(s) x_1^{n,\lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned} \quad (6.42)$$

Let us define a bilinear functional $\mathcal{B}_{n,\lambda_m,t}(\cdot, \cdot)$ on $L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)) \times L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))$ as follows:

$$\begin{aligned} & \mathcal{B}_{n,\lambda_m,t}(v_1, v_2) \\ &= \mathbb{E} \int_t^T \langle Q^{n,\lambda_m}(s) v_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_t^T \langle Q^{n,\lambda_m}(s) x_1^{n,\lambda_m}(s), v_2^n(s) \rangle_{\mathbb{R}^n} ds, \end{aligned} \quad (6.43)$$

$$\forall v_1, v_2 \in L^2_{\mathbb{F}}(t,T;L^4(\Omega;H)).$$

It is easy to check that $\mathcal{B}_{n,\lambda_m,t}(\cdot, \cdot)$ is a bounded bilinear functional. From (6.42), it follows that

$$\begin{aligned} & \mathcal{B}_{n_k^{(4)},\lambda_m,t}(v_1, v_2) \\ &= \mathbb{E} \langle P_T^{n_k^{(4)}} x_1^{n_k^{(4)},\lambda_m}(T), x_2^{n_k^{(4)},\lambda_m}(T) \rangle_{\mathbb{R}^{n_k^{(4)}}} - \mathbb{E} \int_t^T \langle F_{n_k^{(4)}}(s) x_1^{n_k^{(4)},\lambda_m}(s), x_2^{n_k^{(4)},\lambda_m}(s) \rangle_{\mathbb{R}^{n_k^{(4)}}} ds \\ & - \mathbb{E} \int_t^T \langle P^{n_k^{(4)},\lambda_m}(s) K_{n_k^{(4)}}(s) x_1^{n_k^{(4)},\lambda_m}(s), v_2^{n_k^{(4)}}(s) \rangle_{\mathbb{R}^{n_k^{(4)}}} ds \\ & - \mathbb{E} \int_t^T \langle P^{n_k^{(4)},\lambda_m}(s) v_1^{n_k^{(4)}}(s), K_{n_k^{(4)}}(s) x_2^{n_k^{(4)},\lambda_m}(s) + v_2^{n_k^{(4)}}(s) \rangle_{\mathbb{R}^{n_k^{(4)}}} ds. \end{aligned} \quad (6.44)$$

From the definition of P^{λ_m} , $x_1^{n_k^{(4)}}$ and $x_2^{n_k^{(4)}}$, we find that

$$\left\{ \begin{aligned} \lim_{k \rightarrow \infty} \mathbb{E} \langle P_T^{n_k^{(4)}} x_1^{n_k^{(4)}, \lambda_m}(T), x_2^{n_k^{(4)}, \lambda_m}(T) \rangle_{\mathbb{R}^{n_k^{(3)}}} &= \mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H, \\ \lim_{k \rightarrow \infty} \mathbb{E} \int_t^T \langle F_{n_k^{(3)}}(s) x_1^{n_k^{(4)}, \lambda_m}(s), x_2^{n_k^{(4)}, \lambda_m}(s) \rangle_{\mathbb{R}^{n_k^{(4)}}} ds &= \mathbb{E} \int_t^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds, \\ \lim_{k \rightarrow \infty} \mathbb{E} \int_t^T \langle P_{n_k^{(4)}}^{\lambda_m}(s) K_{n_k^{(4)}}(s) x_1^{n_k^{(4)}, \lambda_m}(s), v_2^{n_k^{(4)}}(s) \rangle_{\mathbb{R}^{n_k^{(4)}}} ds &= \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds, \\ \lim_{k \rightarrow \infty} \mathbb{E} \int_t^T \langle P_{n_k^{(4)}}^{\lambda_m}(s) v_1^{n_k^{(4)}}(s), K_{n_k^{(4)}}(s) x_2^{n_k^{(4)}, \lambda_m}(s) + v_2^{n_k^{(4)}}(s) \rangle_{\mathbb{R}^{n_k^{(4)}}} ds \\ &= \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m}(s) + v_2(s) \rangle_H ds, \end{aligned} \right.$$

where x_1 (*resp.* x_2) solves the equation (1.13) (*resp.* (1.14)) with $\xi_1 = 0$ and $u_1 = 0$ (*resp.* $\xi_2 = 0$ and $u_2 = 0$). This, together with (6.44), implies that

$$\begin{aligned} \mathcal{B}_{\lambda_m}^t(v_1, v_2) &\triangleq \lim_{k \rightarrow \infty} \mathcal{B}_{n_k^{(4)}, \lambda_m, t}(v_1, v_2) \\ &= \mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\ &\quad - \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds - \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m}(s) + v_2(s) \rangle_H ds. \end{aligned} \quad (6.45)$$

Noting that the solution of (2.11) (with $\xi_1 = 0$, $u_1 = 0$ and λ replaced by λ_m) satisfies

$$x_1^{\lambda_m}(s) = \int_t^s S_{\lambda_m}(s - \tau) J(\tau) x_1^{\lambda_m}(\tau) d\tau + \int_t^s S_{\lambda_m}(s - \tau) K(\tau) x_1^{\lambda_m}(\tau) d\tau + \int_t^s S_{\lambda_m}(s - \tau) v(\tau) dw(\tau),$$

by means of Lemma 2.1 and Gronwall's inequality, we conclude that

$$|x_1^{\lambda_m}|_{L_{\mathbb{F}}^\infty(t, T; L^4(\Omega; H))} \leq C |v_1|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))}. \quad (6.46)$$

Similarly,

$$|x_2^{\lambda_m}|_{L_{\mathbb{F}}^\infty(t, T; L^4(\Omega; H))} \leq C |v_2|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))}. \quad (6.47)$$

Combining (6.45), (6.46), (6.47) and (6.26), we obtain that

$$|\mathcal{B}_{\lambda_m}^t(v_1, v_2)| \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}) |v_1|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))} |v_2|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))}.$$

Hence, $\mathcal{B}_{\lambda_m}^t(\cdot, \cdot)$ is a bounded bilinear functional on $L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$. Now, for any fixed $v_2 \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$, $\mathcal{B}_{\lambda_m}^t(\cdot, v_2)$ is a bounded linear functional on $L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$.

Therefore, by Lemma 2.3, we can find a unique $\tilde{v}_1 \in L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$ such that

$$\mathcal{B}_{\lambda_m}^t(v_1, v_2) = \langle \tilde{v}_1, v_2 \rangle_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))}, \quad \forall v_2 \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)).$$

Define an operator $\widehat{Q}_3^{\lambda_m, t}$ from $L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ to $L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$ as follows:

$$\widehat{Q}_3^{\lambda_m, t} v_1 = \tilde{v}_1.$$

From the uniqueness of \tilde{v}_1 , it is clear that $\widehat{Q}_3^{\lambda_m, t}$ is well-defined. Further,

$$\begin{aligned} |\widehat{Q}_3^{\lambda_m, t} v_1|_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))} &= |\tilde{v}_1|_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))} \\ &\leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}) |v_1|_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))}. \end{aligned}$$

This shows that $\widehat{Q}_3^{\lambda_m, t}$ is a bounded operator. For any $\alpha, \beta \in \mathbb{R}$ and $v_2, v_3, v_4 \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$,

$$\begin{aligned} &\langle \widehat{Q}_3^{\lambda_m, t}(\alpha v_3 + \beta v_4), v_2 \rangle_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))} \\ &= \mathcal{B}_{\lambda_m}^t(\alpha v_3 + \beta v_4, v_2) = \alpha \mathcal{B}_{\lambda_m}^t(v_3, v_2) + \beta \mathcal{B}_{\lambda_m}^t(v_4, v_2), \end{aligned}$$

which indicates that $\widehat{Q}_3^{\lambda_m, t}(\alpha v_3 + \beta v_4) = \alpha \widehat{Q}_3^{\lambda_m, t} v_3 + \beta \widehat{Q}_3^{\lambda_m, t} v_4$. Hence, $\widehat{Q}_3^{\lambda_m, t}$ is a bounded linear operator from $L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ to $L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$. Put $Q_3^{\lambda_m, t} = \frac{1}{2} \widehat{Q}_3^{\lambda_m, t}$. Then, for any $v_1, v_2 \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$, it holds that

$$\begin{aligned} &\mathcal{B}^t(v_1, v_2) \\ &= \langle Q_3^{\lambda_m, t} v_1, v_2 \rangle_{L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))} + \langle v_1, (Q_3^{\lambda_m, t})^* v_2 \rangle_{L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))}. \end{aligned} \quad (6.48)$$

From (6.20), (6.27), (6.28), (6.34), (6.41), (6.43)–(6.45) and (6.48), we obtain that

$$\begin{aligned} &\mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_{r_j}^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\ &= \mathbb{E} \langle R^{(r_j, \lambda_m)} \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_{r_j}^T \langle P^{\lambda_m}(s) u_1(s), x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_{r_j}^T \langle P^{\lambda_m}(s) x_1^{\lambda_m}(s), u_2(s) \rangle_H ds \\ &\quad + \mathbb{E} \int_{r_j}^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds + \mathbb{E} \int_{r_j}^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m}(s) + v_2(s) \rangle_H ds \\ &\quad + \mathbb{E} \int_{r_j}^T \langle v_1(s), \widehat{Q}_1^{\lambda_m, r_j}(\xi_2)(s) + \widehat{Q}_2^{\lambda_m, r_j}(u_2)(s) + (Q_3^{\lambda_m, r_j})^*(v_2)(s) \rangle_H ds \\ &\quad + \mathbb{E} \int_{r_j}^T \langle Q_1^{\lambda_m, r_j}(\xi_1)(s) + Q_2^{\lambda_m, r_j}(u_1)(s) + Q_3^{\lambda_m, r_j}(v_1)(s), v_2(s) \rangle_H ds, \\ &\quad \forall (\xi_1, u_1, v_1), (\xi_2, u_2, v_2) \in L_{\mathcal{F}_{r_j}}^4(\Omega; H) \times L_{\mathbb{F}}^2(r_j, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(r_j, T; L^4(\Omega; H)), \quad j \in \mathbb{N}. \end{aligned} \quad (6.49)$$

Step 5. In this step, we take $n \rightarrow \infty$ in (6.20) for all $t \in [0, T]$.

Let $u_1 = v_1 = 0$ in (1.13) and $u_2 = v_2 = 0$ in (1.14). By (6.49), we obtain that

$$\begin{aligned} &\mathbb{E} \langle P_T U_{\lambda_m}(T, r_j) \xi_1, U_{\lambda_m}(T, r_j) \xi_2 \rangle_H - \mathbb{E} \int_{r_j}^T \langle F(s) U_{\lambda_m}(s, r_j) \xi_1, U_{\lambda_m}(s, r_j) \xi_2 \rangle_H ds \\ &= \mathbb{E} \langle R^{(r_j, \lambda_m)} \xi_1, \xi_2 \rangle_H. \end{aligned}$$

Hence, for any $\xi_1, \xi_2 \in L_{\mathcal{F}_{r_j}}^4(\Omega; H)$, it holds that

$$\mathbb{E} \left\langle U_{\lambda_m}^*(T, r_j) P_T U_{\lambda_m}(T, r_j) \xi_1 - \int_{r_j}^T U_{\lambda_m}^*(s, r_j) F(s) U_{\lambda_m}(s, r_j) \xi_1 ds, \xi_2 \right\rangle_H = \mathbb{E} \langle R^{(r_j, \lambda_m)} \xi_1, \xi_2 \rangle_H.$$

This leads to

$$\mathbb{E} \left(U_{\lambda_m}^*(T, r_j) P_T U_{\lambda_m}(T, r_j) \xi_1 - \int_{r_j}^T U_{\lambda_m}^*(s, t) F(s) U_{\lambda_m}(s, r_j) \xi_1 ds \mid \mathcal{F}_{r_j} \right) = R^{(r_j, \lambda_m)} \xi_1. \quad (6.50)$$

For any $t \in [0, T]$, $h \in [t, T]$ and $\xi \in L^4_{\mathcal{F}_t}(\Omega; H)$, let us define

$$R^{(h, \lambda_m)} \xi \triangleq \mathbb{E} \left(U_{\lambda_m}^*(T, h) P_T U_{\lambda_m}(T, h) \xi - \int_h^T U_{\lambda_m}^*(s, h) F(s) U_{\lambda_m}(s, h) \xi ds \mid \mathcal{F}_h \right). \quad (6.51)$$

For any $t \leq h_1 \leq h_2 \leq T$ and $\xi \in L^4_{\mathcal{F}_t}(\Omega; H)$, by (6.51), it follows that

$$\begin{aligned} & \mathbb{E} \left| R^{(h_2, \lambda_m)} \xi - R^{(h_1, \lambda_m)} \xi \right|_H^{\frac{4}{3}} \\ & \leq C \left[\mathbb{E} \left| \mathbb{E} \left(U_{\lambda_m}^*(T, h_2) P_T U_{\lambda_m}(T, h_2) \xi - \int_{h_2}^T U_{\lambda_m}^*(s, h_2) F(s) U_{\lambda_m}(s, h_2) \xi ds \mid \mathcal{F}_{h_2} \right) \right. \right. \\ & \quad \left. \left. - \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_2} \right) \right|_H^{\frac{4}{3}} \right. \\ & \quad \left. + \mathbb{E} \left| \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_2} \right) \right. \right. \\ & \quad \left. \left. - \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_1} \right) \right|_H^{\frac{4}{3}} \right]. \end{aligned} \quad (6.52)$$

By Lemma 2.8, it is easy to show that

$$\begin{aligned} & \lim_{h_2 \rightarrow h_1^+} \mathbb{E} \left| \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_2} \right) \right. \\ & \quad \left. - \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_1} \right) \right|_H^{\frac{4}{3}} \\ & = 0. \end{aligned} \quad (6.53)$$

On the other hand,

$$\begin{aligned} & \mathbb{E} \left| \mathbb{E} \left(U_{\lambda_m}^*(T, h_2) P_T U_{\lambda_m}(T, h_2) \xi - \int_{h_2}^T U_{\lambda_m}^*(s, h_2) F(s) U_{\lambda_m}(s, h_2) \xi ds \mid \mathcal{F}_{h_2} \right) \right. \\ & \quad \left. - \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_2} \right) \right|_H^{\frac{4}{3}} \\ & \leq C \mathbb{E} \left| U_{\lambda_m}^*(T, h_2) P_T U_{\lambda_m}(T, h_2) \xi - U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi \right|_H^{\frac{4}{3}} \\ & \quad + C \mathbb{E} \left| \int_{h_2}^T \left[U_{\lambda_m}^*(s, h_2) F(s) U_{\lambda_m}(s, h_2) \xi - U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi \right] ds \right|_H^{\frac{4}{3}} \\ & \quad + C \mathbb{E} \left| \int_{h_1}^{h_2} U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \right|_H^{\frac{4}{3}}. \end{aligned} \quad (6.54)$$

Hence, noting that, for each $m \in \mathbb{N}$, A_{λ_m} is a bounded linear operator on H , we obtain that

$$\begin{aligned} & \lim_{h_2 \rightarrow h_1^+} \mathbb{E} \left| \mathbb{E} \left(U_{\lambda_m}^*(T, h_2) P_T U_{\lambda_m}(T, h_2) \xi - \int_{h_2}^T U_{\lambda_m}^*(s, h_2) F(s) U_{\lambda_m}(s, h_2) \xi ds \mid \mathcal{F}_{h_2} \right) \right. \\ & \quad \left. - \mathbb{E} \left(U_{\lambda_m}^*(T, h_1) P_T U_{\lambda_m}(T, h_1) \xi - \int_{h_1}^T U_{\lambda_m}^*(s, h_1) F(s) U_{\lambda_m}(s, h_1) \xi ds \mid \mathcal{F}_{h_2} \right) \right|_H^{\frac{4}{3}} \\ & = 0. \end{aligned} \quad (6.55)$$

Put

$$\widehat{P}^{\lambda_m}(\cdot) \xi \triangleq R^{(\cdot, \lambda_m)} \xi. \quad (6.56)$$

By (6.52)–(6.53) and (6.55)–(6.56), it is easy to see that $\widehat{P}^{\lambda_m}(\cdot)\xi$ is right continuous in $L^{\frac{4}{3}}_{\mathcal{F}_T}(\Omega; H)$ on $[t, T]$.

For any $t \in [0, T] \setminus \{r_j\}_{j=1}^\infty$, we can find a subsequence $\{r_{j_k}\}_{k=1}^\infty \subset \{r_j\}_{j=1}^\infty$ such that $r_{j_k} > t$ and $\lim_{k \rightarrow \infty} r_{j_k} = t$. Letting $u_1 = v_1 = 0$ in the equation (1.13), and letting $\xi_2 = 0$ and $u_2 = 0$ in the equation (1.14), by (6.49), we find that

$$\begin{aligned} & \mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_{r_{j_k}}^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\ &= \mathbb{E} \int_{r_{j_k}}^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle \chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}(\xi_1)(s), v_2(s) \rangle_H ds. \end{aligned} \quad (6.57)$$

Let us choose $\xi_{1, j_k} \in L^4_{\mathcal{F}_{r_{j_k}}}(\Omega; H)$ such that $|\xi_{1, j_k}|_{L^4_{\mathcal{F}_{r_{j_k}}}(\Omega; H)} = 1$ and

$$\begin{aligned} & |\chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}(\xi_{1, j_k})|_{L^2_{\mathbb{F}}(r_{j_k}, T; L^{\frac{4}{3}}(\Omega; H))} = |\chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}(\xi_{1, j_k})|_{L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H))} \\ & \geq \frac{1}{2} \|\chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}\|_{\mathcal{L}(L^4_{\mathcal{F}_{r_{j_k}}}(\Omega; H), L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)))} \geq \frac{1}{2} \|\chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega; H), L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)))}. \end{aligned} \quad (6.58)$$

Then, we choose $v_{2, j_k} \in L^4_{\mathbb{F}}(r_{j_k}, T; L^4(\Omega; H))$ with $|v_{2, j_k}|_{L^4_{\mathbb{F}}(r_{j_k}, T; L^4(\Omega; H))} = 1$ such that

$$\mathbb{E} \int_t^T \langle \chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}(\xi_{1, j_k})(s), v_{2, j_k}(s) \rangle_H ds \geq \frac{1}{2} |\chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}(\xi_{1, j_k})|_{L^2_{\mathbb{F}}(r_{j_k}, T; L^{\frac{4}{3}}(\Omega; H))}. \quad (6.59)$$

From (6.57)–(6.59), we get that

$$\|\chi_{[r_{j_k}, T]} Q_1^{\lambda_m, r_{j_k}}\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega; H), L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)))} \leq C(\lambda_m) (|P_T|_{L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0, T; \mathcal{L}(H))}),$$

where the constant $C(\lambda_m)$ is independent of r_{j_k} . Similarly,

$$\|\chi_{[r_{j_k}, T]} \widehat{Q}_1^{\lambda_m, r_{j_k}}\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega; H), L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)))} \leq C(\lambda_m) (|P_T|_{L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0, T; \mathcal{L}(H))}).$$

From Lemma 5.1, we conclude that there exist two bounded linear operators $Q_1^{\lambda_m, t}$ and $\widehat{Q}_1^{\lambda_m, t}$, which are from $L^4_{\mathcal{F}_t}(\Omega; H)$ to $L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{j_k^{(1)}\}_{k=1}^\infty \subset \{j_k\}_{k=1}^\infty$ such that

$$\begin{cases} \text{(w)-} \lim_{k \rightarrow \infty} \chi_{[t_{j_k^{(1)}}, T]} Q_1^{\lambda_m, r_{j_k}} \xi = Q_1^{\lambda_m, t} \xi \text{ in } L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)), & \forall \xi \in L^4_{\mathcal{F}_t}(\Omega; H), \\ \text{(w)-} \lim_{k \rightarrow \infty} \chi_{[t_{j_k^{(1)}}, T]} \widehat{Q}_1^{\lambda_m, r_{j_k}} \xi = \widehat{Q}_1^{\lambda_m, t} \xi \text{ in } L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)), & \forall \xi \in L^4_{\mathcal{F}_t}(\Omega; H). \end{cases} \quad (6.60)$$

Letting $\xi_1 = 0$, $v_1 = 0$ in (1.13) and $\xi_2 = 0$, $u_2 = 0$ in (1.14), by (6.49), we obtain that

$$\begin{aligned} & \mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_{r_{j_k}}^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\ &= \mathbb{E} \int_{r_{j_k}}^T \langle P^{\lambda_m}(s) u_1(s), x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_{r_{j_k}}^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds \\ &+ \mathbb{E} \int_t^T \langle \chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}}(u_1)(s), v_2(s) \rangle_H ds, \quad \forall u_1, v_2 \in L^2_{\mathbb{F}}(r_{j_k}, T; L^4(\Omega; H)), \quad k \in \mathbb{N}. \end{aligned} \quad (6.61)$$

We choose $u_1^{(r_{j_k})} \in L_{\mathbb{F}}^2(r_{j_k}, T; L^4(\Omega; H))$ satisfying $|u_1^{(r_{j_k})}|_{L_{\mathbb{F}}^2(r_{j_k}, T; L^4(\Omega; H))} = 1$, and

$$\begin{aligned} & |\chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}}(u_1^{(r_{j_k})})|_{L_{\mathbb{F}}^2(r_{j_k}, T; L^{\frac{4}{3}}(\Omega; H))} \\ & \geq \frac{1}{2} \|\chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}}\|_{\mathcal{L}(L_{\mathbb{F}}^2(r_{j_k}, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(r_{j_k}, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \geq \frac{1}{2} \|\chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}} \circ \chi_{[r_{j_k}, T]}\|_{\mathcal{L}(L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)))}. \end{aligned} \quad (6.62)$$

Then we choose $v_2^{(r_{j_k})} \in L_{\mathbb{F}}^2(r_{j_k}, T; L^4(\Omega; H))$ satisfying $|v_2^{(r_{j_k})}|_{L_{\mathbb{F}}^2(r_{j_k}, T; L^4(\Omega; H))} = 1$, and

$$\mathbb{E} \int_t^T \left\langle \chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}}(u_1^{(r_{j_k})})(s), v_2^{(r_{j_k})}(s) \right\rangle_H ds \geq \frac{1}{2} |\chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}}(u_1^{(r_{j_k})})|_{L_{\mathbb{F}}^2(r_{j_k}, T; L^{\frac{4}{3}}(\Omega; H))}. \quad (6.63)$$

From (6.61)–(6.63), we obtain that

$$\begin{aligned} & \|\chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}} \circ \chi_{[r_{j_k}, T]}\|_{\mathcal{L}(L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(t, T; L^2(\Omega; \mathcal{L}(H)))}), \end{aligned}$$

where the constant $C(\lambda_m)$ is independent of r_{j_k} . By a similar argument, we obtain that

$$\begin{aligned} & \|\chi_{[r_{j_k}, T]} \widehat{Q}_2^{\lambda_m, r_{j_k}} \circ \chi_{[r_{j_k}, T]}\|_{\mathcal{L}(L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(t, T; L^2(\Omega; \mathcal{L}(H)))}). \end{aligned}$$

By means of Lemma 5.1, we see that there exist two bounded linear operators $Q_2^{\lambda_m, t}$ and $\widehat{Q}_2^{\lambda_m, t}$, from $L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ to $L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{r_{j_k}^{(2)}\}_{k=1}^{\infty} \subset \{r_{j_k}^{(1)}\}_{k=1}^{\infty}$ such that, for any $u \in L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$,

$$\begin{cases} \text{(w)-} \lim_{k \rightarrow \infty} \left(\chi_{[r_{j_k}, T]} Q_2^{\lambda_m, r_{j_k}} \circ \chi_{[r_{j_k}, T]} \right) u = Q_2^{\lambda_m, t} u \text{ in } L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)), \\ \text{(w)-} \lim_{k \rightarrow \infty} \left(\chi_{[r_{j_k}, T]} \widehat{Q}_2^{\lambda_m, r_{j_k}} \circ \chi_{[r_{j_k}, T]} \right) u = \widehat{Q}_2^{\lambda_m, t} u \text{ in } L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)). \end{cases} \quad (6.64)$$

For any $t \in [0, T]$, we define two operators $Q^{(\lambda_m, t)}$ and $\widehat{Q}^{(\lambda_m, t)}$ on $L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ as follows:

$$\begin{cases} Q^{(\lambda_m, t)}(\xi, u, v) = Q_1^{\lambda_m, t} \xi + Q_2^{\lambda_m, t} u + Q_3^{\lambda_m, t} v, \\ \widehat{Q}^{(\lambda_m, t)}(\xi, u, v) = \widehat{Q}_1^{\lambda_m, t} \xi + \widehat{Q}_2^{\lambda_m, t} u + (Q_3^{\lambda_m, t})^* v, \\ \forall (\xi, u, v) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)). \end{cases} \quad (6.65)$$

From the definition of $Q_1^{\lambda_m, t}$, $Q_2^{\lambda_m, t}$ and $Q_3^{\lambda_m, t}$ (resp. $\widehat{Q}_1^{\lambda_m, t}$, $\widehat{Q}_2^{\lambda_m, t}$ and $(Q_3^{\lambda_m, t})^*$), we find that $Q^{(\lambda_m, t)}(\cdot, \cdot, \cdot)$ (resp. $\widehat{Q}^{(\lambda_m, t)}(\cdot, \cdot, \cdot)$) is a bounded linear operator from $L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ to $L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$ and $Q^{(\lambda_m, t)}(0, 0, \cdot)^* = \widehat{Q}^{(\lambda_m, t)}(0, 0, \cdot)$.

For any $t \in [0, T]$, from (6.49), (6.56), (6.60), (6.64) and (6.65), we obtain that

$$\begin{aligned}
& \mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\
&= \mathbb{E} \langle \widehat{P}^{\lambda_m}(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) u_1(s), x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) x_1^{\lambda_m}(s), u_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m}(s) + v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(\lambda_m, t)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q^{(\lambda_m, t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds, \\
&\quad \forall (\xi_1, u_1, v_1), (\xi_2, u_2, v_2) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)).
\end{aligned} \tag{6.66}$$

We claim that

$$P^{\lambda_m}(t) \triangleq \widehat{P}^{\lambda_m}(t), \quad \text{a.e. } t \in [0, T]. \tag{6.67}$$

To show this, for any $0 \leq t_1 < t_2 < T$, we choose $x_1(t_1) = \eta_1 \in L_{\mathcal{F}_{t_1}}^4(\Omega; H)$ and $u_1 = v_1 = 0$ in the equation (2.11), and $x_2(t_1) = 0$, $u_2(\cdot) = \frac{\chi_{[t_1, t_2]}}{t_2 - t_1} \eta_2$ with $\eta_2 \in L_{\mathcal{F}_{t_1}}^4(\Omega; H)$ and $v_2 = 0$ in the equation (2.12), by (6.66) and recalling the definition of the evolution operator $U_{\lambda_m}(\cdot, \cdot)$, we see that

$$\begin{aligned}
& \frac{1}{t_2 - t_1} \mathbb{E} \int_{t_1}^{t_2} \langle P^{\lambda_m}(s) U_{\lambda_m}(s, t_1) \eta_1, \eta_2 \rangle_H ds \\
&= \mathbb{E} \langle P_T U_{\lambda_m}(T, t_1) \eta_1, x_{2, t_2}^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_{t_1}^T \langle F(s) U_{\lambda_m}(s, t_1) \eta_1, x_{2, t_2}^{\lambda_m}(s) \rangle_H ds,
\end{aligned} \tag{6.68}$$

where $x_{2, t_2}^{\lambda_m}(\cdot)$ stands for the solution to the equation (2.12) with the above choice of ξ_2 , u_2 and v_2 . It is clear that

$$x_{2, t_2}^{\lambda_m}(s) = \begin{cases} \int_{t_1}^s S_{\lambda_m}(s - \tau) J(\tau) x_{2, t_2}^{\lambda_m}(\tau) d\tau + \int_{t_1}^s S_{\lambda_m}(s - \tau) K(\tau) x_{2, t_2}^{\lambda_m}(\tau) dw(\tau) \\ \quad + \frac{1}{t_2 - t_1} \int_{t_1}^s S_{\lambda_m}(s - \tau) \eta_2 d\tau, & s \in [t_1, t_2], \\ U_{\lambda_m}(s, t_2) x_{2, t_2}^{\lambda_m}(t_2), & s \in [t_2, T]. \end{cases} \tag{6.69}$$

Then, by Lemma 2.1, we see that

$$\begin{aligned}
& \mathbb{E} |x_{2, t_2}^{\lambda_m}(s)|_H^4 \\
& \leq C(\lambda_m) \left\{ \int_{t_1}^s \left[|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 \right] \mathbb{E} |x_{2, t_2}^{\lambda_m}(\tau)|_H^4 d\tau + \mathbb{E} |\eta_2|_H^4 \right\}, \quad \forall s \in [t_1, t_2].
\end{aligned}$$

By Gronwall's inequality, it follows that

$$|x_{2, t_2}^{\lambda_m}|_{L_{\mathbb{F}}^\infty(t_1, t_2; L^4(\Omega; H))} \leq C(\lambda_m) |\eta_2|_{L_{\mathcal{F}_{t_1}}^4(\Omega; H)}, \tag{6.70}$$

where the constant $C(\lambda_m)$ is independent of t_2 . On the other hand, by (6.69), we have

$$\begin{aligned}
& \mathbb{E} |x_{2, t_2}^{\lambda_m}(t_2) - \eta_2|_H^4 \\
& \leq C(\lambda_m) \left\{ \int_{t_1}^s \left[|J(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 \right] \mathbb{E} |x_{2, t_2}^{\lambda_m}(\tau)|_H^4 d\tau \right. \\
& \quad \left. + \mathbb{E} \left| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S(t_2 - \tau) \eta_2 d\tau - \eta_2 \right|_H^4 \right\}.
\end{aligned}$$

This, together with (6.70), implies that

$$\lim_{t_2 \rightarrow t_1 + 0} \mathbb{E} |x_{2,t_2}^{\lambda_m}(t_2) - \eta_2|_H^4 \leq C(\lambda_m) \lim_{t_2 \rightarrow t_1 + 0} \mathbb{E} \left| \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S(t_2 - \tau) \eta_2 d\tau - \eta_2 \right|_H^4 = 0.$$

Therefore, for any $s \in [t_2, T]$,

$$\begin{aligned} & \lim_{t_2 \rightarrow t_1 + 0} \mathbb{E} |U_{\lambda_m}(s, t_2) x_{2,t_2}^{\lambda_m}(t_2) - U_{\lambda_m}(s, t_1) \eta_2|_H^4 \\ & \leq 8 \lim_{t_2 \rightarrow t_1 + 0} \left[\mathbb{E} |U_{\lambda_m}(s, t_2) x_{2,t_2}^{\lambda_m}(t_2) - U_{\lambda_m}(s, t_2) \eta_2|_H^4 + \mathbb{E} |U_{\lambda_m}(s, t_2) \eta_2 - U_{\lambda_m}(s, t_1) \eta_2|_H^4 \right] \\ & \leq C(\lambda_m) \lim_{t_2 \rightarrow t_1 + 0} \left[\mathbb{E} |x_{2,t_2}^{\lambda_m}(t_2) - \eta_2|_H^4 + \mathbb{E} |U_{\lambda_m}(s, t_2) \eta_2 - U_{\lambda_m}(s, t_1) \eta_2|_H^4 \right] = 0. \end{aligned}$$

Hence, we obtain that

$$\lim_{t_2 \rightarrow t_1 + 0} x_{2,t_2}^{\lambda_m}(s) = U_{\lambda_m}(s, t_1) \eta_2 \quad \text{in } L_{\mathcal{F}_s}^4(\Omega; H), \quad \forall s \in [t_2, T]. \quad (6.71)$$

By (6.70) and (6.71), we conclude that

$$\begin{aligned} & \lim_{t_2 \rightarrow t_1 + 0} \left[\mathbb{E} \left\langle P_T U_{\lambda_m}(T, t_1) \eta_1, x_{2,t_2}^{\lambda_m}(T) \right\rangle_H - \mathbb{E} \int_{t_1}^T \left\langle F(s) U_{\lambda_m}(s, t_1) \eta_1, x_{2,t_2}^{\lambda_m}(s) \right\rangle_H ds \right] \\ & = \mathbb{E} \left\langle P_T U_{\lambda_m}(T, t_1) \eta_1, U_{\lambda_m}(T, t_1) \eta_2 \right\rangle_H - \mathbb{E} \int_{t_1}^T \left\langle F(s) U_{\lambda_m}(s, t_1) \eta_1, U_{\lambda_m}(s, t_1) \eta_2 \right\rangle_H ds. \end{aligned} \quad (6.72)$$

By choosing $x_1(t_1) = \eta_1$ and $u_1 = v_1 = 0$ in (1.13), and $x_2(t_1) = \eta_2$ and $u_2 = v_2 = 0$ in (1.14), by (6.66), we find that

$$\begin{aligned} & \mathbb{E} \left\langle \widehat{P}^{\lambda_m}(t_1) \eta_1, \eta_2 \right\rangle_H \\ & = \mathbb{E} \left\langle P_T U_{\lambda_m}(T, t_1) \eta_1, U_{\lambda_m}(T, t_1) \eta_2 \right\rangle_H - \mathbb{E} \int_{t_1}^T \left\langle F(s) U_{\lambda_m}(s, t_1) \eta_1, U_{\lambda_m}(s, t_1) \eta_2 \right\rangle_H ds. \end{aligned} \quad (6.73)$$

Combining (6.68), (6.72) and (6.73), we obtain that

$$\lim_{t_2 \rightarrow t_1 + 0} \frac{1}{t_2 - t_1} \mathbb{E} \int_{t_1}^{t_2} \left\langle P^{\lambda_m}(s) U_{\lambda_m}(s, t_1) \eta_1, \eta_2 \right\rangle_H ds = \mathbb{E} \left\langle \widehat{P}^{\lambda_m}(t_1) \eta_1, \eta_2 \right\rangle_H. \quad (6.74)$$

By Lemma 2.5, we see that there is a monotonically decreasing sequence $\{t_2^{(n)}\}_{n=1}^\infty$ with $t_2^{(n)} > t_1$ for every n , such that

$$\lim_{t_2^{(n)} \rightarrow t_1 + 0} \frac{1}{t_2^{(n)} - t_1} \mathbb{E} \int_{t_1}^{t_2^{(n)}} \left\langle P^{\lambda_m}(s) U_{\lambda_m}(s, t_1) \eta_1, \eta_2 \right\rangle_H ds = \mathbb{E} \left\langle P^{\lambda_m}(t_1) \eta_1, \eta_2 \right\rangle_H, \quad \text{a.e. } t_1 \in [0, T].$$

This, together with (6.74), implies that

$$\mathbb{E} \left\langle \widehat{P}^{\lambda_m}(t_1) \eta_1, \eta_2 \right\rangle_H = \mathbb{E} \left\langle P^{\lambda_m}(t_1) \eta_1, \eta_2 \right\rangle_H, \quad \text{for a.e. } t_1 \in [0, T].$$

Since η_1 and η_2 are arbitrary elements in $L_{\mathcal{F}_{t_1}}^4(\Omega; H)$, we conclude (6.67).

By (6.66) and (6.67), we end up with

$$\begin{aligned}
& \mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\
&= \mathbb{E} \langle P^{\lambda_m}(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) u_1(s), x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) x_1^{\lambda_m}(s), u_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m}(s) + v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(\lambda_m, t)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q^{(\lambda_m, t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds, \\
&\quad \forall (\xi_1, u_1, v_1), (\xi_2, u_2, v_2) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)).
\end{aligned} \tag{6.75}$$

Step 6. In this step, we show the well-posedness of the relaxed transposition solution to (1.10).

Similar to the argument in Steps 4–5, thanks to the uniform estimate (6.14) (with respect to λ_m), we conclude that there exist a subsequence $\{\lambda_{m_j}^{(1)}\}_{j=1}^\infty \subset \{\lambda_m\}_{m=1}^\infty$, a $P(\cdot) \in \mathcal{L}_{pd}(L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)))$, an $R^{(t)} \in \mathcal{L}_{pd}(L_{\mathcal{F}_t}^4(\Omega; H), L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega; H))$, and two bounded linear operators $Q^{(t)}$ and $\widehat{Q}^{(t)}$ from $L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H))$ to $L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H))$ satisfying $Q^{(t)}(0, 0, \cdot)^* = \widehat{Q}^{(t)}(0, 0, \cdot)$, such that

$$(w)\text{-}\lim_{j \rightarrow \infty} P^{\lambda_{m_j}^{(1)}}(\cdot)u(\cdot) = P(\cdot)u(\cdot) \quad \text{in } L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall u(\cdot) \in L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), \tag{6.76}$$

$$(w)\text{-}\lim_{j \rightarrow \infty} P^{\lambda_{m_j}^{(1)}}(t)\xi = R^{(t)}\xi \quad \text{in } L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega; H), \quad \forall \xi \in L_{\mathcal{F}_t}^4(\Omega; H), \tag{6.77}$$

and

$$\left\{ \begin{array}{l} (w)\text{-}\lim_{j \rightarrow \infty} Q^{(\lambda_{m_j}^{(1)}, t)}(\xi, u(\cdot), v(\cdot)) = Q^{(t)}(\xi, u(\cdot), v(\cdot)) \quad \text{in } L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)), \\ (w)\text{-}\lim_{j \rightarrow \infty} \widehat{Q}^{(\lambda_{m_j}^{(1)}, t)}(\xi, u(\cdot), v(\cdot)) = \widehat{Q}^{(t)}(\xi, u(\cdot), v(\cdot)) \quad \text{in } L_{\mathbb{F}}^2(t, T; L^{\frac{4}{3}}(\Omega; H)), \\ \forall (\xi, u(\cdot), v(\cdot)) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)). \end{array} \right. \tag{6.78}$$

By (6.75), and noting (6.76)–(6.78), we find that $(P(\cdot), R(\cdot), Q(\cdot), \widehat{Q}(\cdot))$ satisfies the following variational equality:

$$\begin{aligned}
& \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \\
&= \mathbb{E} \langle R^{(t)} \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s) u_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) x_1(s), u_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) v_1(s), K(s) x_2(s) + v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds, \\
&\quad \forall (\xi_1, u_1, v_1), (\xi_2, u_2, v_2) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)).
\end{aligned} \tag{6.79}$$

Now we show that $R^{(\cdot)}\xi$ is right continuous in $L_{\mathcal{F}_T}^{\frac{4}{3}}(\Omega; H)$ on $[t, T]$ for any $\xi \in L_{\mathcal{F}_t}^4(\Omega; H)$. Since A is usually an unbounded linear operator on H , here we cannot employ the same method for

treating $R^{(\cdot, \lambda_m)}$. Let $(P^n(\cdot), Q^n(\cdot))$ be the transposition solution to (1.10) with the final datum $P_T^n (= \Gamma_n P_T \Gamma_n)$ and the nonhomogeneous term $F_n (= \Gamma_n F \Gamma_n)$. By Theorem 4.2, it follows that

$$P^n(\cdot) \in D_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}_2(H))) \subset D_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H))). \quad (6.80)$$

Further, similar to the derivation of the equality (6.50), we conclude that for any $t \in [0, T]$, $\tau \in [t, T]$ and $\xi \in L^4_{\mathcal{F}_t}(\Omega; H)$, it holds that

$$R^{(\tau)}\xi = \mathbb{E}\left(U^*(T, \tau)P_T U(T, \tau)\xi - \int_{\tau}^T U^*(s, \tau)F(s)U(s, \tau)\xi ds \mid \mathcal{F}_{\tau}\right),$$

and

$$P^n(\tau)\xi = \mathbb{E}\left(U^*(T, \tau)P_T^n U(T, \tau)\xi - \int_{\tau}^T U^*(s, \tau)F_n(s)U(s, \tau)\xi ds \mid \mathcal{F}_{\tau}\right).$$

By (6.80), in order to prove the right continuity of $R^{(\cdot)}\xi$ in $L^{\frac{4}{3}}_{\mathcal{F}_T}(\Omega; H)$, it remains to show that

$$\lim_{n \rightarrow \infty} |R^{(\cdot)}\xi - P^n(\cdot)\xi|_{L^{\infty}_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H))} = 0. \quad (6.81)$$

For this purpose, for any $\tau \in [t, T]$, we see that

$$\begin{aligned} \mathbb{E}|R^{(\tau)}\xi - P^n(\tau)\xi|_{\frac{4}{3}H} &\leq C\mathbb{E}\left|\int_{\tau}^T U^*(s, \tau)[F(s) - F_n(s)]U(s, \tau)\xi ds\right|_{\frac{4}{3}H} \\ &\quad + C\mathbb{E}\left|U^*(T, \tau)(P_T - P_T^n)U(T, \tau)\xi\right|_{\frac{4}{3}H}. \end{aligned}$$

By the first conclusion in Lemma 2.6, we deduce that for any $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ so that for all $\tau \in [t, T]$ and $\tau \leq \sigma \leq \tau + \delta_1$,

$$\mathbb{E}|U(r, \tau)\xi - U(r, \sigma)\xi|_{\frac{4}{3}H} < \varepsilon_1, \quad \forall r \in [\sigma, T]. \quad (6.82)$$

Now, we choose a monotonicity increasing sequence $\{\tau_i\}_{i=1}^{N_1} \subset [0, T]$ for N_1 sufficiently large such that $\tau_{i+1} - \tau_i \leq \delta_1$ with $\tau_1 = t$ and $\tau_{N_1} = T$, and that

$$\left(\int_{\tau_i}^{\tau_{i+1}} \mathbb{E}|F(s)|_{\mathcal{L}(H)}^2 ds\right)^{\frac{2}{3}} < \varepsilon_1, \quad \text{for all } i = 1, \dots, N_1 - 1. \quad (6.83)$$

For any $\tau_i < \tau \leq \tau_{i+1}$, recalling $F_n = \Gamma_n F \Gamma_n$, we conclude that

$$\begin{aligned} &\mathbb{E}\left|\int_{\tau}^T U^*(s, \tau)[F(s) - F_n(s)]U(s, \tau)\xi ds\right|_{\frac{4}{3}H} \\ &\leq C\mathbb{E}\left|\int_{\tau_i}^T U^*(s, \tau)[F(s) - F_n(s)]U(s, \tau_i)\xi ds\right|_{\frac{4}{3}H} + C\mathbb{E}\left|\int_{\tau}^{\tau_i} U^*(s, \tau)[F(s) - F_n(s)]U(s, \tau_i)\xi ds\right|_{\frac{4}{3}H} \\ &\quad + C\mathbb{E}\left|\int_{\tau}^T U^*(s, \tau)[F(s) - F_n(s)][U(s, \tau_i) - U(s, \tau)]\xi ds\right|_{\frac{4}{3}H} \\ &\leq C\int_{\tau}^T \mathbb{E}\left|[F(s) - F_n(s)]U(s, \tau_i)\xi\right|_{\frac{4}{3}H} ds + C\left(\int_{\tau_i}^{\tau_{i+1}} \mathbb{E}|F|^2_{\mathcal{L}(H)} ds\right)^{\frac{2}{3}} \\ &\quad + C\max_{s \in [\tau, T]} \mathbb{E}\left|[U(s, \tau_i) - U(s, \tau)]\xi\right|_{\frac{4}{3}H} ds. \end{aligned} \quad (6.84)$$

By the choice of F_n , there is an integer $N_2(\varepsilon_1) > 0$ so that for all $n > N_2$ and $i = 1, \dots, N_1 - 1$,

$$\int_{\tau_i}^T \mathbb{E} \left| [F(s) - F_n(s)] U(s, \tau_i) \xi \right|_H^{\frac{4}{3}} ds \leq \varepsilon_1. \quad (6.85)$$

Combing (6.82)–(6.85), we conclude that for all $n > N_2$ and $\tau \in [t, T]$,

$$\mathbb{E} \left| \int_{\tau}^T U^*(s, \tau) [F(s) - F_n(s)] U(s, \tau) \xi ds \right|_H^{\frac{4}{3}} \leq C_1 \varepsilon_1. \quad (6.86)$$

Here the constant C_1 is independent of ε_1 , n and τ . Similarly, there is an integer $N_3(\varepsilon_1) > 0$ such that for every $n > N_3$,

$$\mathbb{E} \left| U^*(T, \tau) [P_T - P_T^n] U(T, \tau) \xi \right|_H^{\frac{4}{3}} \leq C_2 \varepsilon_1, \quad (6.87)$$

for the constant C_2 which is independent of ε_1 , n and τ . Now for any $\varepsilon > 0$, let us choose $\varepsilon_1 = \frac{\varepsilon}{C_1 + C_2}$. Then, for all $n > \max\{N_2(\varepsilon_1), N_3(\varepsilon_1)\}$ and $\tau \in [t, T]$,

$$\mathbb{E} |R^{(\cdot)} \xi - P^n(\cdot) \xi|_H^{\frac{4}{3}} < \varepsilon.$$

Therefore, we obtain the desired result (6.81).

Further, similar to (6.67), one can show that

$$P(t) \triangleq R^{(t)} \quad \text{in } L^{\frac{4}{3}}_{\mathcal{F}_t}(\Omega; H), \quad \text{a.e. } t \in [0, T]. \quad (6.88)$$

Combining (6.79) and (6.88), we see that $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)})$ satisfies (6.2). Hence, $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)})$ is a relaxed transposition solution to (1.10), and satisfies the estimate (6.3).

Finally, we show the uniqueness of the relaxed transposition solution to (1.10). Assume that $(\overline{P}(\cdot), \overline{Q}^{(\cdot)}, \overline{\widehat{Q}}^{(\cdot)}) \in D_{\mathbb{F}, w}([0, T]; L^{\frac{4}{3}}(\Omega; \mathcal{L}(H))) \times \mathcal{Q}[0, T]$ is another relaxed transposition solution to the equation (1.10). Then, by Definition 6.1, it follows that

$$\begin{aligned} 0 &= \mathbb{E} \left\langle (\overline{P}(t) - P(t)) \xi_1, \xi_2 \right\rangle_H + \mathbb{E} \int_t^T \left\langle (\overline{P}(s) - P(s)) u_1(s), x_2(s) \right\rangle_H ds \\ &\quad + \mathbb{E} \int_t^T \left\langle (\overline{P}(s) - P(s)) x_1(s), u_2(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle (\overline{P}(s) - P(s)) K(s) x_1(s), v_2(s) \right\rangle_H ds \\ &\quad + \mathbb{E} \int_t^T \left\langle (\overline{P}(s) - P(s)) v_1(s), K(s) x_2(s) + v_2(s) \right\rangle_H ds \\ &\quad + \mathbb{E} \int_t^T \left\langle v_1(s), (\overline{\widehat{Q}}^{(t)} - \widehat{Q}^{(t)}) (\xi_2, u_2, v_2)(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle (\overline{Q}^{(t)} - Q^{(t)}) (\xi_1, u_1, v_1)(s), v_2(s) \right\rangle_H ds, \\ &\quad \forall t \in [0, T]. \end{aligned} \quad (6.89)$$

Choosing $u_1 = u_2 = 0$ and $v_1 = v_2 = 0$ in the test equations (1.13) and (1.14), by (6.89), we obtain that, for any $t \in [0, T]$,

$$0 = \mathbb{E} \left\langle (\overline{P}(t) - P(t)) \xi_1, \xi_2 \right\rangle_H, \quad \forall \xi_1, \xi_2 \in L^{\frac{4}{3}}_{\mathcal{F}_t}(\Omega; H).$$

Hence, we find that $\overline{P}(\cdot) = P(\cdot)$. By this, it is easy to see that (6.89) becomes that

$$\begin{aligned} 0 &= \mathbb{E} \int_t^T \left\langle v_1(s), (\overline{\widehat{Q}}^{(t)} - \widehat{Q}^{(t)}) (\xi_2, u_2, v_2)(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle (\overline{Q}^{(t)} - Q^{(t)}) (\xi_1, u_1, v_1)(s), v_2(s) \right\rangle_H ds, \\ &\quad \forall t \in [0, T]. \end{aligned} \quad (6.90)$$

Choosing $v_2 = 0$ in the test equation (1.14), we see that (6.90) becomes

$$0 = \mathbb{E} \int_t^T \left\langle v_1(s), \left(\overline{\widehat{Q}^{(t)}} - \widehat{Q}^{(t)} \right) (\xi_2, u_2, 0)(s) \right\rangle_H ds. \quad (6.91)$$

Noting that v_1 is arbitrarily in $L_{\mathbb{F}}^2(0, T; L^4(\Omega; H))$, we conclude from (6.91) that $\overline{\widehat{Q}^{(t)}}(\cdot, \cdot, 0) = \widehat{Q}^{(t)}(\cdot, \cdot, 0)$. Similarly, $Q^{(t)}(\cdot, \cdot, 0) = \widehat{Q}^{(t)}(\cdot, \cdot, 0)$. Hence,

$$0 = \mathbb{E} \int_t^T \left\langle v_1(s), \left(\overline{\widehat{Q}^{(t)}} - \widehat{Q}^{(t)} \right) (0, 0, v_2)(s) \right\rangle_H ds + \mathbb{E} \int_t^T \left\langle \left(\overline{Q^{(t)}} - Q^{(t)} \right) (0, 0, v_1)(s), v_2(s) \right\rangle_H ds. \quad (6.92)$$

Since $\overline{Q^{(t)}}(0, 0, \cdot)^* = \overline{\widehat{Q}^{(t)}}(0, 0, \cdot)$ and $Q^{(t)}(0, 0, \cdot)^* = \widehat{Q}^{(t)}(0, 0, \cdot)$, from (6.92), we find that

$$0 = 2\mathbb{E} \int_t^T \left\langle v_1(s), \left(\overline{\widehat{Q}^{(t)}} - \widehat{Q}^{(t)} \right) (0, 0, v_2)(s) \right\rangle_H ds, \quad (6.93)$$

which implies that $\overline{Q^{(t)}}(0, 0, \cdot) = Q^{(t)}(0, 0, \cdot)$ and $\overline{\widehat{Q}^{(t)}}(0, 0, \cdot) = \widehat{Q}^{(t)}(0, 0, \cdot)$. Hence $\overline{Q^{(t)}}(\cdot, \cdot, \cdot) = Q^{(t)}(\cdot, \cdot, \cdot)$ and $\overline{\widehat{Q}^{(t)}}(\cdot, \cdot, \cdot) = \widehat{Q}^{(t)}(\cdot, \cdot, \cdot)$. This completes the proof of Theorem 6.1. \square

Remark 6.2 1) From the variational identity (6.20), it is quite easy to obtain an a priori estimate for P^{n, λ_m} with respect to n (See (6.25)). However, from the same identity, it is clear that Q^{n, λ_m} is not coercive, and therefore, it is very hard to derive any a priori estimate for Q^{n, λ_m} . This is the main obstacle to prove the existence of transposition solution to the equation (1.10) in the general case. As a remedy, we introduce four operators $Q_1^{n, \lambda_m, t}$, $\widehat{Q}_1^{n, \lambda_m, t}$, $Q_2^{n, \lambda_m, t}$ and $\widehat{Q}_2^{n, \lambda_m, t}$ and the bilinear functional $\mathcal{B}_{n, \lambda_m, t}(\cdot, \cdot)$ so that one can obtain suitable a priori estimates and take limit in some sense, and via which we are able to establish the existence of relaxed transposition solution to (1.10) with general data.

2) Alternatively, one may use Theorem 4.2 (instead of Theorem 3.1 (or [18, Theorem 4.1])) to prove Theorem 6.1 (by approximating the data P_T and F respectively by a sequence of $\{P_T^k\}_{k=1}^\infty \subset L_{\mathcal{F}_T}^2(\Omega; \mathcal{L}_2(H))$ and $\{F^k\}_{k=1}^\infty \subset L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}_2(H)))$ in the strong operator topology). Nevertheless, in some sense, our present proof seems to be more close to the numerical approach to solve the equation (1.10).

7 Some properties of the relaxed transposition solutions to the operator-valued BSEs

In this section, we shall derive some properties for the relaxed transposition solutions to the equation (1.10). These properties will play key roles in the proof of our general Pontryagin-type stochastic maximum principle, presented in Section 9.

The following result shows the local Lipschitz continuity of the relaxed transposition solution to (1.10) with respect to its coefficient K .

Theorem 7.1 *Let the assumptions in Theorem 6.1 hold and let $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)})$ be the relaxed transposition solution to (1.10). Let $K^\Delta \in L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))$ and let $(P^\Delta(\cdot), Q^{(\cdot, \Delta)}, \widehat{Q}^{(\cdot, \Delta)})$ be the relaxed transposition solution to the equation (1.10) with K replaced by K^Δ . Then,*

$$\begin{aligned} & \left\| Q^{(0)}(0, 0, \cdot) - Q^{(0, \Delta)}(0, 0, \cdot) \right\|_{\mathcal{L}(L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & + \left\| \widehat{Q}^{(0)}(0, 0, \cdot) - \widehat{Q}^{(0, \Delta)}(0, 0, \cdot) \right\|_{\mathcal{L}(L_{\mathbb{F}}^2(0, T; L^4(\Omega; H)), L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(K^\Delta) \|K - K^\Delta\|_{L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))}. \end{aligned} \quad (7.1)$$

Here the positive constant $C(K^\Delta)$ depends on A , T , $|J|_{L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))}$, $|K|_{L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))}$, $|K^\Delta|_{L^4_{\mathbb{F}}(0,T;L^\infty(\Omega;\mathcal{L}(H)))}$, $|P_T|_{L^2_{\mathcal{F}_T}(\Omega;\mathcal{L}(H))}$ and $|F|_{L^1_{\mathbb{F}}(0,T;L^2(\Omega;\mathcal{L}(H)))}$.

Proof: The proof is divided into several steps.

Step 1. For any $t \in [0, T]$, consider the following two equations:

$$\begin{cases} dx_1^\Delta = (A + J)x_1^\Delta ds + u_1 ds + K^\Delta x_1^\Delta dw(s) + v_1 dw(s) & \text{in } (t, T], \\ x_1^\Delta(t) = \xi_1 \end{cases} \quad (7.2)$$

and

$$\begin{cases} dx_2^\Delta = (A + J)x_2^\Delta ds + u_2 ds + K^\Delta x_2^\Delta dw(s) + v_2 dw(s) & \text{in } (t, T], \\ x_2^\Delta(t) = \xi_2. \end{cases} \quad (7.3)$$

Here $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1, v_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ are the same as that in (1.13)–(1.14). Clearly, for any $s \in [t, T]$, it holds that

$$\begin{aligned} \mathbb{E}|x_1^\Delta(s)|_H^4 &= \mathbb{E}\left|S(s-t)\xi_1 + \int_t^s S(s-\tau)J(\tau)x_1^\Delta(\tau)d\tau + \int_t^s S(s-\tau)u_1(\tau)d\tau \right. \\ &\quad \left. + \int_t^s S(s-\tau)K^\Delta(\tau)x_1^\Delta(\tau)dw + \int_t^s S(s-\tau)v_1(\tau)dw\right|_H^4 \\ &\leq C\left[\mathbb{E}|S(s-t)\xi_1|_H^4 + \mathbb{E}\left|\int_t^s S(s-\tau)J(\tau)x_1^\Delta(\tau)d\tau\right|_H^4 + \mathbb{E}\left|\int_t^s S(s-\tau)u_1(\tau)d\tau\right|_H^4 \right. \\ &\quad \left. + \mathbb{E}\left|\int_t^s S(s-\tau)K^\Delta(\tau)x_1^\Delta(\tau)dw\right|_H^4 + \mathbb{E}\left|\int_t^s S(s-\tau)v_1(\tau)dw\right|_H^4\right]. \end{aligned} \quad (7.4)$$

By Lemma 2.1, it is easy to see that

$$\begin{aligned} \mathbb{E}\left|\int_t^s S(s-\tau)K^\Delta(\tau)x_1^\Delta(\tau)dw\right|_H^4 &\leq C\mathbb{E}\left[\int_t^s |S(s-\tau)K^\Delta(\tau)x_1^\Delta(\tau)|_H^2 d\tau\right]^2 \\ &\leq C\int_t^s \mathbb{E}\left[|K^\Delta(\tau)x_1^\Delta(\tau)|_H\right]^4 d\tau \leq C\int_t^s |K^\Delta(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4 \mathbb{E}|x_1^\Delta(\tau)|_H^4 d\tau. \end{aligned}$$

This, together with (7.4), implies that

$$\begin{aligned} \mathbb{E}|x_1^\Delta(s)|_H^4 &\leq C\left[|\xi_1|_{L^4_{\mathcal{F}_0}(\Omega;H)}^4 + |u_1|_{L^2_{\mathbb{F}}(0,T;L^4(\Omega;H))}^4 + |v_1|_{L^2_{\mathbb{F}}(0,T;L^4(\Omega;H))}^4\right] \\ &\quad + C\int_t^s [|J(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4 + |K^\Delta(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4] \mathbb{E}|x_1^\Delta(\tau)|_H^4 d\tau. \end{aligned} \quad (7.5)$$

By Gronwall's inequality, we obtain that

$$|x_1^\Delta|_{L^\infty(t,T;L^4(\Omega;H))} \leq C(K^\Delta)(|\xi_1|_{L^4_{\mathcal{F}_t}(\Omega;H)} + |u_1|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))} + |v_1|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))}). \quad (7.6)$$

Similarly,

$$|x_2^\Delta|_{L^\infty(t,T;L^4(\Omega;H))} \leq C(K^\Delta)(|\xi_2|_{L^4_{\mathcal{F}_t}(\Omega;H)} + |u_2|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))} + |v_2|_{L^2_{\mathbb{F}}(t,T;L^4(\Omega;H))}). \quad (7.7)$$

Let $y_1^\Delta = x_1 - x_1^\Delta$ and $y_2^\Delta = x_2 - x_2^\Delta$. From (1.13) and (7.2), we see that y_1^Δ solves

$$\begin{cases} dy_1^\Delta = (A + J)y_1^\Delta ds + Ky_1^\Delta dw(s) + (K - K^\Delta)x_1^\Delta dw(s) & \text{in } (t, T], \\ y_1^\Delta(t) = 0. \end{cases} \quad (7.8)$$

Then, similar to (7.5) and by (7.6), we have

$$\begin{aligned}
& \mathbb{E}|y_1^\Delta(s)|_H^4 \\
& \leq C|(K - K^\Delta)x_1^\Delta|_{L_{\mathbb{F}}^2(0,T;L^4(\Omega;H))}^4 + C \int_t^s [|J(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4] \mathbb{E}|y_1^\Delta(\tau)|_H^4 d\tau \\
& \leq C(K^\Delta)|K - K^\Delta|_{L_{\mathbb{F}}^4(0,T;L^\infty(\Omega;\mathcal{L}(H)))}^4 (|\xi_1|_{L_{\mathcal{F}_t}^4(\Omega;H)} + |u_1|_{L_{\mathbb{F}}^2(t,T;L^4(\Omega;H))} + |v_1|_{L_{\mathbb{F}}^2(t,T;L^4(\Omega;H))})^4 \\
& \quad + C \int_t^s [|J(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4 + |K(\tau)|_{L^\infty(\Omega;\mathcal{L}(H))}^4] \mathbb{E}|y_1^\Delta(\tau)|_H^4 d\tau.
\end{aligned}$$

This, together with the Gronwall's inequality, implies that

$$\begin{aligned}
& \sup_{s \in [t,T]} \mathbb{E}|x_1(s) - x_1^\Delta(s)|_H = \sup_{s \in [t,T]} \mathbb{E}|y_1^\Delta(s)|_H \\
& \leq C(K^\Delta)|K - K^\Delta|_{L_{\mathbb{F}}^4(0,T;L^\infty(\Omega;\mathcal{L}(H)))}^4 (|\xi_1|_{L_{\mathcal{F}_t}^4(\Omega;H)} + |u_1|_{L_{\mathbb{F}}^2(t,T;L^4(\Omega;H))} + |v_1|_{L_{\mathbb{F}}^2(t,T;L^4(\Omega;H))}).
\end{aligned} \tag{7.9}$$

Similarly,

$$\begin{aligned}
& \sup_{s \in [t,T]} \mathbb{E}|x_2(s) - x_2^\Delta(s)|_H \\
& \leq C(K^\Delta)|K - K^\Delta|_{L_{\mathbb{F}}^4(0,T;L^\infty(\Omega;\mathcal{L}(H)))}^4 (|\xi_2|_{L_{\mathcal{F}_t}^4(\Omega;H)} + |u_2|_{L_{\mathbb{F}}^2(t,T;L^4(\Omega;H))} + |v_2|_{L_{\mathbb{F}}^2(t,T;L^4(\Omega;H))}).
\end{aligned} \tag{7.10}$$

Step 2. By Definition 6.1, it follows that

$$\begin{aligned}
& \mathbb{E}\langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E}\langle P_T x_1^\Delta(T), x_2^\Delta(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s)x_1(s), x_2(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle F(s)x_1^\Delta(s), x_2^\Delta(s) \rangle_H ds \\
& = \mathbb{E}\langle (P(t) - P^\Delta(t))\xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s)u_1(s), x_2(s) \rangle_H ds - \mathbb{E} \int_t^T \langle P^\Delta(s)u_1(s), x_2^\Delta(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle P(s)x_1(s), u_2(s) \rangle_H ds - \mathbb{E} \int_t^T \langle P^\Delta(s)x_1^\Delta(s), u_2(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle P(s)K(s)x_1(s), v_2(s) \rangle_H ds - \mathbb{E} \int_t^T \langle P^\Delta(s)K^\Delta(s)x_1^\Delta(s), v_2(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle P(s)v_1(s), K(s)x_2(s) + v_2(s) \rangle_H ds - \mathbb{E} \int_t^T \langle P^\Delta(s)v_1(s), K^\Delta(s)x_2^\Delta(s) + v_2(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s) \rangle_H ds - \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(t,\Delta)}(\xi_2, u_2, v_2)(s) \rangle_H ds \\
& + \mathbb{E} \int_t^T \langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds - \mathbb{E} \int_t^T \langle Q^{(t,\Delta)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds, \\
& \quad \forall (\xi_1, u_1, v_1), (\xi_2, u_2, v_2) \in L_{\mathcal{F}_t}^4(\Omega; H) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)) \times L_{\mathbb{F}}^2(t, T; L^4(\Omega; H)).
\end{aligned} \tag{7.11}$$

Letting $u_1 = u_2 = v_1 = v_2 = 0$ in the test equations (1.13) and (1.14), respectively, from (7.11),

we find that

$$\begin{aligned}
& \mathbb{E} \langle (P(t) - P^\Delta(t)) \xi_1, \xi_2 \rangle_H \\
&= \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \langle P_T x_1^\Delta(T), x_2^\Delta(T) \rangle_H \\
&\quad - \mathbb{E} \int_0^T \langle F(s) x_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle F(s) x_1^\Delta(s), x_2^\Delta(s) \rangle_H ds \\
&= \mathbb{E} \langle P_T [x_1(T) - x_1^\Delta(T)], x_2(T) \rangle_H + \mathbb{E} \langle P_T x_1^\Delta(T), x_2(T) - x_2^\Delta(T) \rangle_H \\
&\quad - \mathbb{E} \int_0^T \langle F(s) [x_1(s) - x_1^\Delta(s)], x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle F(s) x_1^\Delta(s), x_2(s) - x_2^\Delta(s) \rangle_H ds.
\end{aligned} \tag{7.12}$$

In (7.12), we choose $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$ with $|\xi_1|_{L^4_{\mathcal{F}_t}(\Omega; H)} = |\xi_2|_{L^4_{\mathcal{F}_t}(\Omega; H)} = 1$, such that

$$\mathbb{E} \langle (P(t) - P^\Delta(t)) \xi_1, \xi_2 \rangle_H \geq \frac{1}{2} \|P(t) - P^\Delta(t)\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega; H), L^{\frac{4}{3}}_{\mathcal{F}_t}(\Omega; H))}.$$

On the other hand, by (7.6)–(7.7) and (7.9)–(7.10), we have

$$\begin{aligned}
& \left| \mathbb{E} \langle P_T [x_1(T) - x_1^\Delta(T)], x_2(T) \rangle_H + \mathbb{E} \langle P_T x_1^\Delta(T), x_2(T) - x_2^\Delta(T) \rangle_H \right. \\
& \quad \left. - \mathbb{E} \int_0^T \langle F(s) [x_1(s) - x_1^\Delta(s)], x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle F(s) x_1^\Delta(s), x_2(s) - x_2^\Delta(s) \rangle_H ds \right| \\
& \leq C(K^\Delta) |P_T|_{L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))} (|x_1(T) - x_1^\Delta(T)|_{L^4_{\mathcal{F}_T}(\Omega; H)} + |x_2(T) - x_2^\Delta(T)|_{L^4_{\mathcal{F}_T}(\Omega; H)}) \\
& \quad + C(K^\Delta) |F|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))} (|x_1 - x_1^\Delta|_{L^\infty_{\mathbb{F}}(t, T; L^4(\Omega; H))} + |x_2 - x_2^\Delta|_{L^\infty_{\mathbb{F}}(t, T; L^4(\Omega; H))}) \\
& \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}.
\end{aligned} \tag{7.13}$$

Hence,

$$\|P(t) - P^\Delta(t)\|_{\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega; H), L^{\frac{4}{3}}_{\mathcal{F}_t}(\Omega; H))} \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}, \quad \forall t \in [0, T]. \tag{7.14}$$

Step 3. Letting $\xi_1 = \xi_2 = 0$ and $u_1 = u_2 = 0$ in the test equations (1.13) and (1.14) respectively, from (7.11) and noting that

$$\mathbb{E} \int_0^T \langle v_1(s), \widehat{Q}^{(t)}(0, 0, v_2)(s) \rangle_H ds = \mathbb{E} \int_0^T \langle Q^{(t)}(0, 0, v_1)(s), v_2(s) \rangle_H ds$$

and

$$\mathbb{E} \int_0^T \langle v_1(s), \widehat{Q}^{(0, \Delta)}(0, 0, v_2)(s) \rangle_H ds = \mathbb{E} \int_0^T \langle Q^{(0, \Delta)}(0, 0, v_1)(s), v_2(s) \rangle_H ds,$$

we find that, for any $v_1, v_2 \in L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))$, it holds that

$$\begin{aligned}
& \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \langle P_T x_1^\Delta(T), x_2^\Delta(T) \rangle_H - \mathbb{E} \int_0^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \\
& \quad + \mathbb{E} \int_0^T \langle F(s) x_1^\Delta(s), x_2^\Delta(s) \rangle_H ds \\
&= \mathbb{E} \int_0^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds - \mathbb{E} \int_0^T \langle P^\Delta(s) K^\Delta(s) x_1^\Delta(s), v_2(s) \rangle_H ds \\
& \quad + \mathbb{E} \int_0^T \langle P(s) v_1(s), K(s) x_2(s) + v_2(s) \rangle_H ds - \mathbb{E} \int_0^T \langle P^\Delta(s) v_1(s), K^\Delta(s) x_2^\Delta(s) + v_2(s) \rangle_H ds \\
& \quad + 2\mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, v_1)(s), v_2(s) \rangle_H ds - 2\mathbb{E} \int_0^T \langle Q^{(0, \Delta)}(0, 0, v_1)(s), v_2(s) \rangle_H ds.
\end{aligned} \tag{7.15}$$

We choose $v_1, v_2 \in L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))$ with $|v_1|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))} = |v_2|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))} = 1$, such that

$$\begin{aligned} & 2\mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, v_1)(s), v_2(s) \rangle_H ds - 2\mathbb{E} \int_0^T \langle Q^{(0, \Delta)}(0, 0, v_1)(s), v_2(s) \rangle_H ds \\ & \geq \|Q^{(0)}(0, 0, \cdot) - Q^{(0, \Delta)}(0, 0, \cdot)\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))}. \end{aligned} \quad (7.16)$$

By the above choice of v_1 and v_2 , similar to (7.13), we have

$$\begin{aligned} & \left| \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \langle P_T x_1^\Delta(T), x_2^\Delta(T) \rangle_H - \mathbb{E} \int_0^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \right. \\ & \quad \left. + \mathbb{E} \int_0^T \langle F(s) x_1^\Delta(s), x_2^\Delta(s) \rangle_H ds \right| \\ & \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}. \end{aligned} \quad (7.17)$$

By (7.14), it follows that

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds - \mathbb{E} \int_0^T \langle P^\Delta(s) K^\Delta(s) x_1^\Delta(s), v_2(s) \rangle_H ds \right| \\ & \leq \left| \mathbb{E} \int_0^T \langle P(s) K(s) [x_1(s) - x_1^\Delta(s)], v_2(s) \rangle_H ds \right| + \left| \mathbb{E} \int_0^T \langle P(s) [K(s) - K^\Delta(s)] x_1^\Delta(s), v_2(s) \rangle_H ds \right| \\ & \quad + \left| \mathbb{E} \int_0^T \langle [P(s) - P^\Delta(s)] K^\Delta(s) x_1^\Delta(s), v_2(s) \rangle_H ds \right| \\ & \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}. \end{aligned} \quad (7.18)$$

Similarly,

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \langle P(s) v_1(s), K(s) x_2(s) + v_2(s) \rangle_H ds - \mathbb{E} \int_0^T \langle P^\Delta(s) v_1(s), K^\Delta(s) x_2^\Delta(s) + v_2(s) \rangle_H ds \right| \\ & \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}. \end{aligned} \quad (7.19)$$

From (7.15)–(7.19), we obtain that

$$\begin{aligned} & \|Q^{(0)}(0, 0, \cdot) - Q^{(0, \Delta)}(0, 0, \cdot)\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}. \end{aligned}$$

Similarly,

$$\begin{aligned} & \|\widehat{Q}^{(0)}(0, 0, \cdot) - \widehat{Q}^{(0, \Delta)}(0, 0, \cdot)\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(K^\Delta) |K - K^\Delta|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))}. \end{aligned}$$

Hence, we obtain the desired estimate (7.1). This completes the proof of Theorem 7.1. \square

Next, we shall show a property of the relaxed transposition solution to (1.10) when the coefficient K is piecewisely constant with respect to the time variable. For this purpose, we introduce the following subspace of $L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))$ (Recall (5.2) for the definition of \mathcal{M}):

$$\mathcal{H} = \left\{ \sum_{i=1}^{\ell} \chi_{O_i}(\cdot) h_i \mid \ell \in \mathbb{N}, O_i \in \mathcal{M}, h_i \in D(A) \right\}. \quad (7.20)$$

It is clear that \mathcal{H} is dense in $L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))$. We have the following result.

Theorem 7.2 Suppose that the assumptions in Theorem 6.1 hold and $K = \sum_{i=1}^{n_0} \chi_{[t_i, t_{i+1})}(t) K_i$ for some $n_0 \in \mathbb{N}$, $0 = t_1 < t_2 < \dots < t_{n_0} < t_{n_0+1} = T$, and $K_i \in L_{\mathcal{F}_{t_i}}^\infty(\Omega; \mathcal{L}(D(A)))$, $i = 1, \dots, n_0$. Let $(P(\cdot), Q(\cdot), \widehat{Q}(\cdot))$ be the relaxed transposition solution to (1.10). Then, there exist two pointwisely defined linear operators Q and \widehat{Q} , both of which are from \mathcal{H} to $L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H))$, such that

$$\begin{aligned} & \mathbb{E} \int_0^T \langle v_1(s), \widehat{Q}^{(0)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_0^T \langle Q^{(0)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds \\ &= \mathbb{E} \int_0^T \left[\langle (Qv_1)(s), x_2(s) \rangle_H + \langle x_1(s), (\widehat{Q}v_2)(s) \rangle_H \right] ds, \end{aligned} \quad (7.21)$$

holds for any $\xi_1, \xi_2 \in L_{\mathcal{F}_0}^4(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L_{\mathbb{F}}^2(0, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in \mathcal{H}$. Here, $x_1(\cdot)$ and $x_2(\cdot)$ solve accordingly (1.13) and (1.14) with $t = 0$.

Proof: As in the proof of Theorem 6.1 (but with the set $\{r_j\}_{j=1}^\infty$ (introduced at the very beginning of Step 4) being replaced by $\{r_j\}_{j=1}^\infty \cup \{t_1, t_2, \dots, t_{n_0}\}$), we introduce the equation (6.4) (approximating to the equation (1.10)), and the equations (6.8) and (6.9) (which are accordingly finite approximations of the equations (1.13) and (1.14)), and obtain the approximate variational equality (6.20) for $P^{n, \lambda}(\cdot)$ and $Q^{n, \lambda}(\cdot)$. Also, we fix a sequence $\{\lambda_m\}_{m=1}^\infty \subset \rho(A)$ such that $\lambda_m \rightarrow \infty$ as $m \rightarrow \infty$. We divide the rest of proof into two steps.

Step 1. Choose $\xi_2^n = 0$ and $u_2^n = 0$ in the equation (6.9). Then, there is a constant $C_1(\lambda_m) > 0$, independent of t and n , such that $|x_2^{n, \lambda}|_{L^\infty(t, T; L^4(\Omega; H))} \leq C_1(\lambda_m) |v_2^n|_{L^2(t, T; L^4(\Omega; H))}$. Without loss of generality, we may assume that

$$\frac{1}{\max_{1 \leq i \leq n_0} (t_{i+1} - t_i)} > 2^{12} |C_1(\lambda_m)|^4 |K|_{L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))}^4. \quad (7.22)$$

(Otherwise, we may choose a refined partition of $[0, T]$ so that (7.22) holds). Letting $\xi_1 \in L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A))$, $u_1^n = -(A^{n, \lambda_m} + J_n)\xi_1$ and $v_1^n = -K_n\xi_1$ in (6.8), and letting $\xi_2^n = 0$ and $u_2^n = 0$ in (6.9), by (6.20) with $t = t_{n_0}$, we find that

$$\begin{aligned} & \mathbb{E} \langle P_T^n \xi_1, x_2^{n, \lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_{t_{n_0}}^T \langle F_n(s) \xi_1, x_2^{n, \lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \int_{t_{n_0}}^T \langle P^{n, \lambda_m}(s) u_1^n(s), x_2^{n, \lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0}}^T \langle P^{n, \lambda_m}(s) K_n(s) \xi_1, v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_{t_{n_0}}^T \langle P^{n, \lambda_m}(s) v_1^n(s), K_n(s) x_2^{n, \lambda_m} + v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &- \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n, \lambda_m}(s) K_n(s) \xi_1, x_2^{n, \lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n, \lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned} \quad (7.23)$$

First, we find a $\xi_1 \in L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A))$ with $|\xi_1|_{L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A))} = 1$ such that

$$|Q^{n, \lambda_m}(\cdot) \xi_1^n|_{L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H))} \geq \frac{1}{2} \|Q^{n, \lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))}.$$

Next, we find a $v_2 \in L_{\mathbb{F}}^2(t_{n_0}, T; L^4(\Omega; H))$ with $|v_2|_{L_{\mathbb{F}}^2(t_{n_0}, T; L^4(\Omega; H))} = 1$ so that

$$\mathbb{E} \int_{t_{n_0}}^T \langle Q^{n, \lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds \geq \frac{1}{2} |Q^{n, \lambda_m}(\cdot) \xi_1^n|_{L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H))}.$$

Hence,

$$\mathbb{E} \int_{t_{n_0}}^T \langle Q^{n,\lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds \geq \frac{1}{4} \|Q^{n,\lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))}. \quad (7.24)$$

On the other hand, by (7.22), it follows that

$$\begin{aligned} & \left| \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n,\lambda_m}(s) K_n \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \right| \\ & \leq |x_2^{n,\lambda_m}|_{L_{\mathbb{F}}^\infty(t_{n_0}, T; L^4(\Omega; H))} \int_{t_{n_0}}^T |Q^{n,\lambda_m}(s) K_n \xi_1|_{L_{\mathcal{F}_s}^{\frac{4}{3}}(\Omega; H)} ds \\ & \leq \sqrt{T - t_{n_0}} |x_2^{n,\lambda_m}|_{L_{\mathbb{F}}^\infty(t_{n_0}, T; L^4(\Omega; H))} |K_{n_0}|_{L_{\mathcal{F}_{t_{n_0}}}^\infty(\Omega; \mathcal{L}(D(A)))} \|Q^{n,\lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq (t_{n_0+1} - t_{n_0})^{\frac{1}{4}} C_1(\lambda_m) |K|_{L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))} \|Q^{n,\lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq \frac{1}{8} \|Q^{n,\lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))}. \end{aligned}$$

This, together with (7.24), implies that

$$\begin{aligned} & \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n,\lambda_m}(s) K_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n,\lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ & \geq \frac{1}{8} \|Q^{n,\lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))}. \end{aligned} \quad (7.25)$$

On the other hand, from the choice of ξ_1 , u_1^n and v_1^n , we find that

$$|u_1^n|_{L_{\mathbb{F}}^2(0, T; L^4(\Omega; H))} \leq C(\lambda_m) (|A|_{\mathcal{L}(D(A), H)} + |J|_{L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H)))}).$$

Hence, by the estimate (6.25), it follows that

$$\begin{aligned} & \left| \mathbb{E} \langle P_T^n \xi_1, x_2^{n,\lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_{t_{n_0}}^T \langle F_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds - \mathbb{E} \int_{t_{n_0}}^T \langle P^{n,\lambda_m}(s) u_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \right. \\ & \quad \left. - \mathbb{E} \int_{t_{n_0}}^T \langle K_n(s) \xi_1, P^{n,\lambda_m}(s) v_2^n(s) \rangle_{\mathbb{R}^n} ds - \mathbb{E} \int_{t_{n_0}}^T \langle P^{n,\lambda_m}(s) v_1^n(s), K_n(s) x_2^{n,\lambda_m} + v_2^n(s) \rangle_{\mathbb{R}^n} ds \right| \\ & \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^4(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}) (1 + |A|_{\mathcal{L}(D(A), H)} + |(J, K)|_{(L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H))))^2}). \end{aligned} \quad (7.26)$$

Combining (7.23) and (7.25)–(7.26), we find that

$$\begin{aligned} & \|Q^{n,\lambda_m}(\cdot)\|_{\mathcal{L}(L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)), L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(\lambda_m) (|P_T|_{L_{\mathcal{F}_T}^4(\Omega; \mathcal{L}(H))} + |F|_{L_{\mathbb{F}}^1(0, T; L^2(\Omega; \mathcal{L}(H)))}) (1 + |A|_{\mathcal{L}(D(A), H)} + |(J, K)|_{(L_{\mathbb{F}}^4(0, T; L^\infty(\Omega; \mathcal{L}(H))))^2}). \end{aligned} \quad (7.27)$$

By (7.27) and Corollary 5.1, there exist a bounded, pointwisely defined linear operator $Q_{t_{n_0}}^{\lambda_m}$ from $L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A))$ to $L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{n_k^{(5)}\}_{k=1}^\infty$ of $\{n_k^{(4)}\}_{k=1}^\infty$ (defined between (6.40) and (6.41)) such that

$$(w)\text{-}\lim_{k \rightarrow \infty} Q_{t_{n_0}}^{n_k^{(5)}, \lambda_m} \xi = Q_{t_{n_0}}^{\lambda_m} \xi \quad \text{in } L_{\mathbb{F}}^2(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall \xi \in L_{\mathcal{F}_{t_{n_0}}}^4(\Omega; D(A)). \quad (7.28)$$

Next, letting $\xi_1 \in L^4_{\mathcal{F}_{t_{n_0-1}}}(\Omega; D(A))$, $u_1^n = -(A^{n,\lambda_m} + J_n)\xi_1$ and $v_1^n = -K_n\xi_1$ in (6.8), and letting $\xi_2 = 0$ and $u_2^n = 0$ in (6.9), by (6.20) with $t = t_{n_0-1}$, we find that

$$\begin{aligned} & \mathbb{E}\langle P_T^n \xi_1, x_2^{n,\lambda}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_{t_{n_0-1}}^T \langle F_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \int_{t_{n_0-1}}^T \langle P^{n,\lambda}(s) u_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0-1}}^T \langle P^{n,\lambda_m}(s) K_n(s) \xi_1, v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_{t_{n_0-1}}^T \langle P^{n,\lambda_m}(s) v_1^n(s), K_n(s) x_2^{n,\lambda_m} + v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_{t_{n_0-1}}^T \langle Q^{n,\lambda_m}(s) K_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0-1}}^T \langle Q^{n,\lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned} \quad (7.29)$$

On the other hand, for these data ξ_1 , u_1^n , v_1^n , ξ_2 , u_2^n and v_2^n , from the variational equality (6.20) with $t = t_{n_0}$, we obtain that

$$\begin{aligned} & \mathbb{E}\langle P_T^n \xi_1, x_2^{n,\lambda_m}(T) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_{t_{n_0}}^T \langle F_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E}\langle P^{n,\lambda_m}(t_{n_0}) \xi_1, x_2^{n,\lambda_m}(t_{n_0}) \rangle_{\mathbb{R}^n} + \mathbb{E} \int_{t_{n_0}}^T \langle P^{n,\lambda_m}(s) u_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_{t_{n_0}}^T \langle P^{n,\lambda_m}(s) K_n(s) \xi_1, v_2^n(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0}}^T \langle P^{n,\lambda_m}(s) v_1^n(s), K_n(s) x_2^{n,\lambda_m} + v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n,\lambda_m}(s) K_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n,\lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned} \quad (7.30)$$

From (7.29) and (7.30), it follows that

$$\begin{aligned} & \mathbb{E}\langle P^{n,\lambda_m}(t_{n_0}) \xi_1, x_2^{n,\lambda_m}(t_{n_0}) \rangle_{\mathbb{R}^n} - \mathbb{E} \int_{t_{n_0-1}}^{t_{n_0}} \langle F_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds \\ &= \mathbb{E} \int_{t_{n_0-1}}^{t_{n_0}} \langle P^{n,\lambda_m}(s) u_1^n(s), x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0-1}}^{t_{n_0}} \langle P^{n,\lambda_m}(s) K_n(s) \xi_1, v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &+ \mathbb{E} \int_{t_{n_0-1}}^{t_{n_0}} \langle P^{n,\lambda_m}(s) v_1^n(s), K_n(s) x_2^{n,\lambda_m} + v_2^n(s) \rangle_{\mathbb{R}^n} ds \\ &- \mathbb{E} \int_{t_{n_0-1}}^{t_{n_0}} \langle Q^{n,\lambda_m}(s) K_n(s) \xi_1, x_2^{n,\lambda_m}(s) \rangle_{\mathbb{R}^n} ds + \mathbb{E} \int_{t_{n_0-1}}^{t_{n_0}} \langle Q^{n,\lambda_m}(s) \xi_1^n, v_2^n(s) \rangle_{\mathbb{R}^n} ds. \end{aligned} \quad (7.31)$$

Similar to (7.27), from (7.31), one obtains that

$$\begin{aligned} & \|Q^{n,\lambda_m}(\cdot)\|_{L(L^4_{\mathcal{F}_{t_{n_0-1}}}(\Omega; D(A)), L^2_{\mathbb{F}}(t_{n_0-1}, t_{n_0}; L^{\frac{4}{3}}(\Omega; H)))} \\ & \leq C(\lambda_m) (|P_T|_{L^4_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))}) (1 + |A|_{\mathcal{L}(D(A), H)} + |(J, K)|_{(L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H))))^2}). \end{aligned} \quad (7.32)$$

By (7.32) and utilizing Corollary 5.1, we conclude that there exist a bounded, pointwisely defined linear operator $Q^{\lambda_m}_{t_{n_0-1}}$ from $L^4_{\mathcal{F}_{t_{n_0-1}}}(\Omega; D(A))$ to $L^2_{\mathbb{F}}(t_{n_0-1}, t_{n_0}; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{n_k^{(6)}\}_{k=1}^\infty$ of $\{n_k^{(5)}\}_{k=1}^\infty$ such that

$$(w)\text{-}\lim_{k \rightarrow \infty} Q^{n_k^{(6)}, \lambda_m} \xi = Q^{\lambda_m}_{t_{n_0-1}} \xi \quad \text{in } L^2_{\mathbb{F}}(t_{n_0-1}, t_{n_0}; L^{\frac{4}{3}}(\Omega; H)), \quad \forall \xi \in L^4_{\mathcal{F}_{t_i}}(\Omega; D(A)). \quad (7.33)$$

Generally, for any $i = 1, 2, \dots, n_0$, there exist a bounded, pointwisely defined linear operator $Q_{t_i}^{\lambda_m}$ from $L_{\mathcal{F}_{t_i}}^4(\Omega; D(A))$ to $L_{\mathbb{F}}^2(t_i, t_{i+1}; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{n_k^{(5+n_0-i)}\}_{k=1}^\infty$ of $\{n_k^{(4+n_0-i)}\}_{k=1}^\infty$ such that

$$(w)\text{-}\lim_{k \rightarrow \infty} Q_{t_i}^{n_k^{(5+n_0-i)}, \lambda_m} \xi = Q_{t_i}^{\lambda_m} \xi \quad \text{in } L_{\mathbb{F}}^2(t_i, t_{i+1}; L^{\frac{4}{3}}(\Omega; H)), \quad \forall \xi \in L_{\mathcal{F}_{t_i}}^4(\Omega; D(A)). \quad (7.34)$$

Since $Q_{t_i}^{\lambda_m}$ is pointwisely defined, for a.e. $(t, \omega) \in (t_i, t_{i+1}) \times \Omega$, there is a $q_{t_i}^{\lambda_m}(t, \omega) \in \mathcal{L}(D(A), H)$ such that

$$(Q_{t_i}^{\lambda_m} \xi)(t, \omega) = q_{t_i}^{\lambda_m}(t, \omega) \xi(\omega), \quad \forall \xi \in L_{\mathcal{F}_{t_i}}^4(\Omega; D(A)).$$

For each $i = 1, 2, \dots, n_0$, write

$$\mathcal{H}_i = \left\{ \sum_{j=1}^{\ell} \chi_{O_j \cap ([t_i, T] \times \Omega)}(\cdot) h_j \mid \ell \in \mathbb{N}, O_j \in \mathcal{M}, h_j \in L_{\mathcal{F}_{t_i}}^4(\Omega; D(A)) \right\}.$$

It is clear that \mathcal{H}_i is dense in $L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H))$ and $\mathcal{H} \subset \mathcal{H}_1$. Define an operator Q^{i, λ_m} from \mathcal{H}_i to $L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H))$ as follows: For any $v = \sum_{j=1}^{\ell} \chi_{O_j \cap ([t_i, T] \times \Omega)}(\cdot) h_j \in \mathcal{H}_i$ with $\ell \in \mathbb{N}$, $O_j \in \mathcal{M}$ and $h_j \in L_{\mathcal{F}_{t_i}}^4(\Omega; D(A))$,

$$(Q^{i, \lambda_m} v)(t, \omega) = \sum_{\gamma=i}^{n_0} \sum_{j=1}^{\ell} \chi_{[t_\gamma, t_{\gamma+1})}(t) \chi_{O_j \cap ([t_i, T] \times \Omega)}(t, \omega) q_{t_\gamma}^{\lambda_m}(t, \omega) h_j, \quad \text{a.e. } (t, \omega) \in (0, T) \times \Omega.$$

It is easy to check that $Q^{i, \lambda_m} v \in L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H))$, and Q^{i, λ_m} is a pointwisely defined linear operator from \mathcal{H}_i to $L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H))$. Also, for the above v , we have

$$Q_{n_k}^{n_k^{(n_0+4)}, \lambda_m}(s) v_{n_k}^{n_k^{(n_0+4)}}(s) = \sum_{j=1}^{\ell} \chi_{O_j \cap ([t_i, T] \times \Omega)} Q_{n_k}^{n_k^{(n_0+4)}, \lambda_m}(s) \Gamma_{n_k^{(n_0+4)}} h_j.$$

Hence,

$$\begin{aligned} & Q_{n_k}^{n_k^{(n_0+4)}, \lambda_m}(\cdot) v_{n_k}^{n_k^{(n_0+4)}}(\cdot) - (Q^{i, \lambda_m} v)(\cdot) \\ &= \sum_{j=1}^{\ell} \chi_{O_j \cap ([t_i, T] \times \Omega)}(\cdot) \left[Q_{n_k}^{n_k^{(n_0+4)}, \lambda_m}(\cdot) \Gamma_{n_k^{(n_0+4)}} h_j - (Q^{i, \lambda_m} h_j)(\cdot) \right]. \end{aligned}$$

This gives that

$$(w)\text{-}\lim_{k \rightarrow \infty} Q_{n_k}^{n_k^{(n_0+4)}, \lambda_m}(\cdot) v_{n_k}^{n_k^{(n_0+4)}}(\cdot) = Q^{i, \lambda_m} v \quad \text{in } L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall v \in \mathcal{H}_i. \quad (7.35)$$

Similarly, one can find a subsequence $\{n_k^{(n_0+5)}\}_{k=1}^\infty \subset \{n_k^{(n_0+4)}\}_{k=1}^\infty$ and a pointwisely defined linear operator $\widehat{Q}^{i, \lambda_m}$ from \mathcal{H}_i to $L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H))$ such that

$$(w)\text{-}\lim_{k \rightarrow \infty} Q_{n_k}^{n_k^{(n_0+5)}, \lambda_m}(\cdot) v_{n_k}^{n_k^{(n_0+5)}}(\cdot) = \widehat{Q}^{i, \lambda_m} v \quad \text{in } L_{\mathbb{F}}^2(t_i, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall v \in \mathcal{H}_i. \quad (7.36)$$

For any $v_1, v_2 \in \mathcal{H}_i$, we have

$$\begin{aligned}
\mathbb{E} \int_{t_i}^T \langle Q^{i, \lambda_m} v_1, v_2 \rangle_H dt &= \lim_{k \rightarrow \infty} \mathbb{E} \int_{t_i}^T \langle Q^{n_k^{(n_0+5)}, \lambda_m}(t) v_1^{n_k^{(n_0+5)}}(t), v_2^{n_k^{(n_0+5)}}(t) \rangle_H dt \\
&= \lim_{k \rightarrow \infty} \mathbb{E} \int_{t_i}^T \langle v_1^{n_k^{(n_0+5)}}(t), Q^{n_k^{(n_0+5)}, \lambda_m}(t)^* v_2^{n_k^{(n_0+5)}}(t) \rangle_H dt \\
&= \mathbb{E} \int_{t_i}^T \langle v_1, \widehat{Q}^{i, \lambda_m} v_2 \rangle_H dt.
\end{aligned} \tag{7.37}$$

For any $\xi_1, \xi_2 \in L^4_{\mathcal{F}_{t_i}}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t_i, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in \mathcal{H}_i$, by (7.35)–(7.36), it is easy to see that

$$\begin{aligned}
&\lim_{k \rightarrow \infty} \mathbb{E} \int_{t_i}^T \left[\langle Q^{n_k^{(n_0+5)}, \lambda_m}(s) v_1^{n_k^{(n_0+5)}}(s), x_2^{n_k^{(n_0+5)}, \lambda_m}(s) \rangle_{\mathbb{R}^{n_k^{(n_0+5)}}} \right. \\
&\quad \left. + \langle Q^{n_k^{(n_0+5)}, \lambda_m}(s) x_1^{n_k^{(n_0+5)}, \lambda_m}(s), v_2^{n_k^{(n_0+5)}}(s) \rangle_{\mathbb{R}^{n_k^{(n_0+5)}}} \right] ds \\
&= \mathbb{E} \int_{t_i}^T \left[\langle (Q^{i, \lambda_m} v_1)(s), x_2(s) \rangle_H + \langle x_1(s), (\widehat{Q}^{i, \lambda_m} v_2)(s) \rangle_H \right] ds.
\end{aligned} \tag{7.38}$$

Therefore,

$$\begin{aligned}
&\mathbb{E} \int_{t_i}^T \langle v_1(s), \widehat{Q}^{(t_i, \lambda_m)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_{t_i}^T \langle Q^{(t_i, \lambda_m)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds \\
&= \mathbb{E} \int_{t_i}^T \left[\langle (Q^{i, \lambda_m} v_1)(s), x_2(s) \rangle_H + \langle x_1(s), (\widehat{Q}^{i, \lambda_m} v_2)(s) \rangle_H \right] ds.
\end{aligned} \tag{7.39}$$

By (6.75) and (7.39), we find that

$$\begin{aligned}
&\mathbb{E} \langle P_T x_1^{\lambda_m}(T), x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_{t_i}^T \langle F(s) x_1^{\lambda_m}(s), x_2^{\lambda_m}(s) \rangle_H ds \\
&= \mathbb{E} \langle P^{\lambda_m}(0) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_{t_i}^T \langle P^{\lambda_m}(s) u_1(s), x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_{t_i}^T \langle P^{\lambda_m}(s) x_1^{\lambda_m}(s), u_2(s) \rangle_H ds \\
&\quad + \mathbb{E} \int_{t_i}^T \langle P^{\lambda_m}(s) K(s) x_1^{\lambda_m}(s), v_2(s) \rangle_H ds + \mathbb{E} \int_{t_i}^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m}(s) + v_2(s) \rangle_H ds \\
&\quad + \mathbb{E} \int_{t_i}^T \left[\langle (Q^{i, \lambda_m} v_1)(s), x_2^{\lambda_m}(s) \rangle_H + \langle x_1^{\lambda_m}(s), (\widehat{Q}^{i, \lambda_m} v_2)(s) \rangle_H \right] ds,
\end{aligned} \tag{7.40}$$

holds for any $\xi_1, \xi_2 \in L^4_{\mathcal{F}_{t_i}}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t_i, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in \mathcal{H}_i$ ($i = 1, 2, \dots, n_0$).

Step 2. We now take $m \rightarrow \infty$ in (7.40). The argument below is very similar to Step 1. Choose $\xi_2 = 0$ and $u_2 = 0$ in the equation (2.12). By (2.14) and similar to (6.14), there is a constant $C_2 > 0$, independent of t and m , such that $|x_2^{\lambda_m}|_{L^\infty(t, T; L^4(\Omega; H))} \leq C_2 |v_2|_{L^2(t, T; L^4(\Omega; H))}$. Without loss of generality, we may assume that

$$\frac{1}{\max_{1 \leq i \leq n_0} (t_{i+1} - t_i)} > 2^{12} |C_2|^4 |K|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))}^4. \tag{7.41}$$

Letting $\xi_1 \in L^4_{\mathcal{F}_{t_{n_0}}}(\Omega; D(A))$, $u_1 = -(A^{\lambda_m} + J)\xi_1$ and $v_1 = -K\xi_1$ in (2.11), and letting $\xi_2 = 0$ and $u_2 = 0$ in (2.12), by (7.40) and (7.37), we find that

$$\begin{aligned}
& \mathbb{E} \langle P_T \xi_1, x_2^{\lambda_m}(T) \rangle_H - \mathbb{E} \int_{t_{n_0}}^T \langle F(s) \xi_1, x_2^{\lambda_m}(s) \rangle_H ds \\
&= \mathbb{E} \int_{t_{n_0}}^T \langle P^{\lambda_m}(s) u_1(s), x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_{t_{n_0}}^T \langle P^{\lambda_m}(s) K(s) \xi_1, v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_{t_{n_0}}^T \langle P^{\lambda_m}(s) v_1(s), K(s) x_2^{\lambda_m} + v_2(s) \rangle_H ds \\
&- \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n_0, \lambda_m} K(s) \xi_1, x_2^{\lambda_m}(s) \rangle_H ds + \mathbb{E} \int_{t_{n_0}}^T \langle Q^{n_0, \lambda_m} \xi_1, v_2(s) \rangle_H ds.
\end{aligned} \tag{7.42}$$

Similar to (7.27), by (7.42) and noting (7.41), we have the following estimate:

$$\begin{aligned}
& \|Q^{n_0, \lambda_m}(\cdot)\|_{\mathcal{L}(L^4_{\mathcal{F}_{t_{n_0}}}(\Omega; D(A)), L^2_{\mathbb{F}}(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)))} \\
& \leq C(|P_T|_{L^4_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))} + |F|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))}) (1 + |A|_{\mathcal{L}(D(A), H)} + |(J, K)|_{(L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H))))^2}).
\end{aligned} \tag{7.43}$$

By (7.43) and Corollary 5.1, there exist a bounded, pointwisely defined linear operator $Q_{t_{n_0}}$ from $L^4_{\mathcal{F}_{t_{n_0}}}(\Omega; D(A))$ to $L^2_{\mathbb{F}}(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{\lambda_{m_j}^{(2)}\}_{j=1}^\infty$ of $\{\lambda_{m_j}^{(1)}\}_{j=1}^\infty$ (defined at the beginning of the Step 6 in the proof of Theorem 6.1) such that

$$(w)\text{-} \lim_{j \rightarrow \infty} Q^{n_0, \lambda_{m_j}^{(2)}} \xi = Q_{t_{n_0}} \xi \quad \text{in } L^2_{\mathbb{F}}(t_{n_0}, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall \xi \in L^4_{\mathcal{F}_{t_{n_0}}}(\Omega; D(A)).$$

Generally, for any $i = 1, 2, \dots, n_0$, there exist a bounded, pointwisely defined linear operator Q_{t_i} from $L^4_{\mathcal{F}_{t_i}}(\Omega; D(A))$ to $L^2_{\mathbb{F}}(t_i, t_{i+1}; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{\lambda_m^{(n_0-i+2)}\}_{m=1}^\infty$ of $\{\lambda_m^{(n_0-i+1)}\}_{m=1}^\infty$ such that

$$(w)\text{-} \lim_{j \rightarrow \infty} Q^{i, \lambda_{m_j}^{(n_0-i+2)}} \xi = Q_{t_i} \xi \quad \text{in } L^2_{\mathbb{F}}(t_i, t_{i+1}; L^{\frac{4}{3}}(\Omega; H)), \quad \forall \xi \in L^4_{\mathcal{F}_{t_i}}(\Omega; D(A)).$$

Since Q_{t_i} is pointwisely defined, for a.e. $(t, \omega) \in (t_i, t_{i+1}) \times \Omega$, there is a $q_{t_i}(t, \omega) \in \mathcal{L}(D(A), H)$ such that

$$(Q_{t_i} \xi)(t, \omega) = q_{t_i}(t, \omega) \xi(\omega), \quad \forall \xi \in L^4_{\mathcal{F}_{t_i}}(\Omega; D(A)).$$

Define a linear operator Q from \mathcal{H} to $L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))$ as follows: For any $v = \sum_{j=1}^\ell \chi_{O_i} h_i \in \mathcal{H}$ with $\ell \in \mathbb{N}$, $O_i \in \mathcal{M}$ and $h_i \in D(A)$,

$$(Qv)(t, \omega) = \sum_{\gamma=1}^{n_0} \sum_{j=1}^\ell \chi_{[t_\gamma, t_{\gamma+1})}(t) \chi_{O_j}(t, \omega) q_{t_\gamma}(t, \omega) h_j, \quad \text{a.e. } (t, \omega) \in (0, T) \times \Omega.$$

Then,

$$(w)\text{-} \lim_{j \rightarrow \infty} Q^{1, \lambda_{m_j}^{(n_0+1)}} v = Qv \quad \text{in } L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall v \in \mathcal{H}. \tag{7.44}$$

By a similar argument, we see that there exist a linear operator \widehat{Q} from \mathcal{H} to $L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))$, and a subsequence $\{\lambda_{m_j}^{(n_0+2)}\}_{j=1}^{\infty}$ of $\{\lambda_{m_j}^{(n_0+1)}\}_{j=1}^{\infty}$ such that

$$(w)\text{-} \lim_{j \rightarrow \infty} \widehat{Q}^{1, \lambda_{m_j}^{(n_0+2)}} v = \widehat{Q}v \quad \text{in } L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)), \quad \forall v \in \mathcal{H}. \quad (7.45)$$

Now, choosing arbitrarily $\xi_1, \xi_2 \in H$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in \mathcal{H}$, by (7.38) and (7.44)–(7.45), we find that

$$\begin{aligned} & \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \mathbb{E} \int_0^T \left[\langle Q^{n_k^{(2n_0+4)}, \lambda_{m_j}^{(n_0+2)}}(s) v_1^{n_k^{(2n_0+4)}}(s), x_2^{n_k^{(2n_0+4)}, \lambda_{m_j}^{(n_0+2)}}(s) \rangle_{\mathbb{R}^{n_k^{(2n_0+4)}}} \right. \\ & \quad \left. + \langle Q^{n_k^{(2n_0+4)}, \lambda_{m_j}^{(n_0+2)}}(s) x_1^{n_k^{(2n_0+4)}, \lambda_{m_j}^{(n_0+2)}}(s), v_2^{n_k^{(2n_0+4)}}(s) \rangle_{\mathbb{R}^{n_k^{(2n_0+4)}}} \right] ds \\ & = \mathbb{E} \int_0^T \left[\langle (Qv_1)(s), x_2(s) \rangle_H + \langle x_1(s), (\widehat{Q}v_2)(s) \rangle_H \right] ds. \end{aligned} \quad (7.46)$$

Combining (6.48), (6.60), (6.64), (6.65), (6.78) and (7.46), we conclude that the desired equality (7.21) holds for any $\xi_1, \xi_2 \in L^4_{\mathcal{F}_0}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in \mathcal{H}$. This completes the proof of Theorem 7.2. \square

Remark 7.1 1) We conjecture that the same conclusion in Theorem 7.2 still holds for any $K \in L^4_{\mathbb{F}}(0, T; L^{\infty}(\Omega; \mathcal{L}(H)))$, or at least for any $K = \sum_{i=1}^{n_0} \chi_{[t_i, t_{i+1})}(t) K_i$ with $n_0 \in \mathbb{N}$, $0 = t_1 < t_2 < \dots < t_{n_0} < t_{n_0+1} = T$, and $K_i \in L^{\infty}_{\mathcal{F}_{t_i}}(\Omega; \mathcal{L}(H))$ (If the later is true, then we may drop the assumption $b_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^4_{\mathbb{F}}(0, T; L^{\infty}(\Omega; \mathcal{L}(D(A))))$ in Theorem 9.1). However, we cannot prove it at this moment.

2) In some sense, the operators Q and \widehat{Q} given in Theorem 7.2 play similar roles as the operators Q and Q^* , where the later operator Q is the second component of the transposition solution $(P(\cdot), Q(\cdot))$ to (1.10).

8 Necessary condition for optimal controls, the case of convex control domains

For the sake of completeness, in this section, we shall give a necessary condition for optimal controls of the system (1.2) for the case of special control domain U , i.e., U is a convex subset of another separable Hilbert space H_1 , and the metric of U is introduced by the norm of H_1 (i.e., $d(u_1, u_2) = |u_1 - u_2|_{H_1}$).

To begin with, we introduce the following further assumptions for $a(\cdot, \cdot, \cdot)$, $b(\cdot, \cdot, \cdot)$, $g(\cdot, \cdot, \cdot)$ and $h(\cdot)$.

(A3) The maps $a(t, x, u)$ and $b(t, x, u)$, and the functional $g(t, x, u)$ and $h(x)$ are C^1 with respect to x and u . Moreover, there exists a constant $C_L > 0$ such that, for any $(t, x, u) \in [0, T] \times H \times U$,

$$\begin{cases} \|a_x(t, x, u)\|_{\mathcal{L}(H)} + \|b_x(t, x, u)\|_{\mathcal{L}(H)} + |g_x(t, x, u)|_H + |h_x(x)|_H \leq C_L, \\ \|a_u(t, x, u)\|_{\mathcal{L}(H_1, H)} + \|b_u(t, x, u)\|_{\mathcal{L}(H_1, H)} + |g_u(t, x, u)|_{H_1} \leq C_L. \end{cases} \quad (8.1)$$

Our result in this section is as follows.

Theorem 8.1 Assume that $x_0 \in L^2_{\mathbb{F}_0}(\Omega; H)$. Let the assumptions (A1), (A2) and (A3) hold, and let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (P). Let $(y(\cdot), Y(\cdot))$ be the transposition solution of the equation (1.8) with $p = 2$, and y_T and $f(\cdot, \cdot, \cdot)$ given by

$$\begin{cases} y_T = -h_x(\bar{x}(T)), \\ f(t, y_1, y_2) = -a_x(t, \bar{x}(t), \bar{u}(t))^* y_1 - b_x(t, \bar{x}(t), \bar{u}(t))^* y_2 + g_x(t, \bar{x}(t), \bar{u}(t)). \end{cases} \quad (8.2)$$

Then,

$$\begin{aligned} \operatorname{Re} \langle a_u(t, \bar{x}(t), \bar{u}(t))^* y(t) + b_u(t, \bar{x}(t), \bar{u}(t))^* Y(t) - g_u(t, \bar{u}(t), \bar{x}(t)), u - \bar{u}(t) \rangle_{H_1} &\leq 0, \\ \text{a.e. } [0, T] \times \Omega, \forall u \in U. \end{aligned} \quad (8.3)$$

Proof: We use the convex perturbation technique and divide the proof into several steps.

Step 1. For the optimal pair $(\bar{x}(\cdot), \bar{u}(\cdot))$, we fix arbitrarily a control $u(\cdot) \in \mathcal{U}[0, T]$ satisfying $u(\cdot) - \bar{u}(\cdot) \in L^2_{\mathbb{F}}(0, T; L^2(\Omega; H_1))$. Since U is convex, we see that

$$u^\varepsilon(\cdot) = \bar{u}(\cdot) + \varepsilon[u(\cdot) - \bar{u}(\cdot)] = (1 - \varepsilon)\bar{u}(\cdot) + \varepsilon u(\cdot) \in \mathcal{U}[0, T], \quad \forall \varepsilon \in [0, 1].$$

Denote by $x^\varepsilon(\cdot)$ the state process of (1.2) corresponding to the control $u^\varepsilon(\cdot)$. By Lemma 1.1, it follows that

$$|x^\varepsilon|_{C_{\mathbb{F}}(0, T; L^2(\Omega; H))} \leq C(1 + |x_0|_{L^2_{\mathbb{F}_0}(\Omega; H)}), \quad \forall \varepsilon \in [0, 1]. \quad (8.4)$$

Write $x_1^\varepsilon(\cdot) = \frac{1}{\varepsilon}[x^\varepsilon(\cdot) - \bar{x}(\cdot)]$ and $\delta u(\cdot) = u(\cdot) - \bar{u}(\cdot)$. Since $(\bar{x}(\cdot), \bar{u}(\cdot))$ satisfies (1.2), it is easy to see that $x_1^\varepsilon(\cdot)$ satisfies the following stochastic differential equation:

$$\begin{cases} dx_1^\varepsilon = (Ax_1^\varepsilon + a_1^\varepsilon x_1^\varepsilon + a_2^\varepsilon \delta u)dt + (b_1^\varepsilon x_1^\varepsilon + b_2^\varepsilon \delta u)dw(t) & \text{in } (0, T], \\ x_1^\varepsilon(0) = 0, \end{cases} \quad (8.5)$$

where

$$\begin{cases} a_1^\varepsilon(t) = \int_0^1 a_x(t, \bar{x}(t) + \sigma \varepsilon x_1^\varepsilon(t), u^\varepsilon(t))d\sigma, & a_2^\varepsilon(t) = \int_0^1 a_u(t, \bar{x}(t), \bar{u}(t) + \sigma \varepsilon \delta u(t))d\sigma, \\ b_1^\varepsilon(t) = \int_0^1 b_x(t, \bar{x}(t) + \sigma \varepsilon x_1^\varepsilon(t), u^\varepsilon(t))d\sigma, & b_2^\varepsilon(t) = \int_0^1 b_u(t, \bar{x}(t), \bar{u}(t) + \sigma \varepsilon \delta u(t))d\sigma. \end{cases} \quad (8.6)$$

Consider the following stochastic differential equation:

$$\begin{cases} dx_2 = [Ax_2 + a_1(t)x_2 + a_2(t)\delta u]dt + [b_1(t)x_2 + b_2(t)\delta u]dw(t) & \text{in } (0, T], \\ x_2(0) = 0, \end{cases} \quad (8.7)$$

where

$$\begin{cases} a_1(t) = a_x(t, \bar{x}(t), \bar{u}(t)), & a_2(t) = a_u(t, \bar{x}(t), \bar{u}(t)), \\ b_1(t) = b_x(t, \bar{x}(t), \bar{u}(t)), & b_2(t) = b_u(t, \bar{x}(t), \bar{u}(t)). \end{cases} \quad (8.8)$$

Step 2. In this step, we shall show that

$$\lim_{\varepsilon \rightarrow 0+} |x_1^\varepsilon - x_2|_{L^\infty_{\mathbb{F}}(0, T; L^2(\Omega; H))} = 0. \quad (8.9)$$

First, using Lemma 2.1 and by the assumption (A1), we find that

$$\begin{aligned}
\mathbb{E}|x_1^\varepsilon(t)|_H^2 &= \mathbb{E} \left| \int_0^t S(t-s)a_1^\varepsilon(s)x_1^\varepsilon(s)ds + \int_0^t S(t-s)a_2^\varepsilon(s)\delta u(s)ds \right. \\
&\quad \left. + \int_0^t S(t-s)b_1^\varepsilon(s)x_1^\varepsilon(s)dw(s) + \int_0^t S(t-s)b_2^\varepsilon(s)\delta u(s)dw(s) \right|_H^2 \\
&\leq C\mathbb{E} \left[\left| \int_0^t S(t-s)a_1^\varepsilon(s)x_1^\varepsilon(s)ds \right|_H^2 + \left| \int_0^t S(t-s)b_1^\varepsilon(s)x_1^\varepsilon(s)dw(s) \right|_H^2 \right. \\
&\quad \left. + \left| \int_0^t S(t-s)a_2^\varepsilon(s)\delta u(s)ds \right|_H^2 + \left| \int_0^t S(t-s)b_2^\varepsilon(s)\delta u(s)dw(s) \right|_H^2 \right] \\
&\leq C \left[\int_0^t \mathbb{E}|x_1^\varepsilon(s)|_H^2 ds + \int_0^T \mathbb{E}|\delta u(s)|_{H_1}^2 dt \right].
\end{aligned} \tag{8.10}$$

It follows from (8.10) and Gronwall's inequality that

$$\mathbb{E}|x_1^\varepsilon(t)|_H^2 \leq C|\bar{u} - u|_{L_{\mathbb{F}}^2(0,T;L^2(\Omega;H_1))}^2, \quad \forall t \in [0, T]. \tag{8.11}$$

By a similar computation, we see that

$$\mathbb{E}|x_2(t)|_H^2 \leq C|\bar{u} - u|_{L_{\mathbb{F}}^2(0,T;L^2(\Omega;H_1))}^2, \quad \forall t \in [0, T]. \tag{8.12}$$

On the other hand, put $x_3^\varepsilon = x_1^\varepsilon - x_2$. Then, x_3^ε solves the following equation:

$$\begin{cases} dx_3^\varepsilon = [Ax_3^\varepsilon + a_1^\varepsilon(t)x_3^\varepsilon + (a_1^\varepsilon(t) - a_1(t))x_2 + (a_2^\varepsilon(t) - a_2(t))\delta u]dt \\ \quad + [b_1^\varepsilon(t)x_3^\varepsilon + (b_1^\varepsilon(t) - b_1(t))x_2 + (b_2^\varepsilon(t) - b_2(t))\delta u]dw(t) & \text{in } (0, T], \\ x_3^\varepsilon(0) = 0. \end{cases} \tag{8.13}$$

It follows from (8.12)–(8.13) that

$$\begin{aligned}
&\mathbb{E}|x_3^\varepsilon(t)|_H^2 \\
&= \mathbb{E} \left| \int_0^t S(t-s)a_1^\varepsilon(s)x_3^\varepsilon(s)ds + \int_0^t S(t-s)b_1^\varepsilon(s)x_3^\varepsilon(s)dw(s) \right. \\
&\quad + \int_0^t S(t-s)[a_1^\varepsilon(s) - a_1(s)]x_2(s)ds + \int_0^t S(t-s)[b_1^\varepsilon(s) - b_1(s)]x_2(s)dw(s) \\
&\quad \left. + \int_0^t S(t-s)[a_2^\varepsilon(s) - a_2(s)]\delta u(s)ds + \int_0^t S(t-s)[b_2^\varepsilon(s) - b_2(s)]\delta u(s)dw(s) \right|_H^2 \\
&\leq C \left[\mathbb{E} \int_0^t |x_3^\varepsilon(s)|_H^2 ds \right. \\
&\quad + |x_2(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^2(\Omega;H))}^2 \int_0^T \mathbb{E}(\|a_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)}^2 + \|b_1^\varepsilon(s) - b_1(s)\|_{\mathcal{L}(H)}^2) dt \\
&\quad + |u - \bar{u}|_{L_{\mathbb{F}}^2(0,T;L^2(\Omega;H_1))}^2 \int_0^T \mathbb{E}(\|a_2^\varepsilon(s) - a_2(s)\|_{\mathcal{L}(H_1,H)}^2 + \|b_2^\varepsilon(s) - b_2(s)\|_{\mathcal{L}(H_1,H)}^2) dt \Big] \\
&\leq C(1 + |u - \bar{u}|_{L_{\mathbb{F}}^2(0,T;L^2(\Omega;H_1))}^2) \left\{ \mathbb{E} \int_0^t |x_3^\varepsilon(s)|_H^2 ds + \int_0^T \mathbb{E} \left[\|a_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)}^2 \right. \right. \\
&\quad \left. \left. + \|b_1^\varepsilon(s) - b_1(s)\|_{\mathcal{L}(H)}^2 + \|a_2^\varepsilon(s) - a_2(s)\|_{\mathcal{L}(H_1,H)}^2 + \|b_2^\varepsilon(s) - b_2(s)\|_{\mathcal{L}(H_1,H)}^2 \right] dt \right\}.
\end{aligned}$$

This, together with Gronwall's inequality, implies that

$$\begin{aligned}
\mathbb{E}|x_3^\varepsilon(t)|_H^2 &\leq C e^{C|u - \bar{u}|_{L_{\mathbb{F}}^2(0,T;L^2(\Omega;H_1))}} \int_0^T \mathbb{E} \left[\|a_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)}^2 + \|b_1^\varepsilon(s) - b_1(s)\|_{\mathcal{L}(H)}^2 \right. \\
&\quad \left. + \|a_2^\varepsilon(s) - a_2(s)\|_{\mathcal{L}(H_1,H)}^2 + \|b_2^\varepsilon(s) - b_2(s)\|_{\mathcal{L}(H_1,H)}^2 \right] ds, \quad \forall t \in [0, T].
\end{aligned} \tag{8.14}$$

Note that (8.11) implies $x^\varepsilon(\cdot) \rightarrow \bar{x}(\cdot)$ (in H) in probability, as $\varepsilon \rightarrow 0$. Hence, by (8.6), (8.8) and the continuity of $a_x(t, \cdot, \cdot)$, $b_x(t, \cdot, \cdot)$, $a_u(t, \cdot, \cdot)$ and $b_u(t, \cdot, \cdot)$, we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \mathbb{E} \left[\|a_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)}^2 + \|b_1^\varepsilon(s) - b_1(s)\|_{\mathcal{L}(H)}^2 + \|a_2^\varepsilon(s) - a_2(s)\|_{\mathcal{L}(H_1, H)}^2 + \|b_2^\varepsilon(s) - b_2(s)\|_{\mathcal{L}(H_1, H)}^2 \right] ds = 0.$$

This, combined with (8.14), gives (8.9).

Step 3. Since $(\bar{x}(\cdot), \bar{u}(\cdot))$ is an optimal pair of Problem (P), from (8.9), we find that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot))}{\varepsilon} \\ &= \text{Re} \left\{ \mathbb{E} \int_0^T \left[\langle g_1(t), x_2(t) \rangle_H + \langle g_2(t), \delta u(t) \rangle_{H_1} \right] dt + \mathbb{E} \langle h_x(\bar{x}(T)), x_2(T) \rangle_H \right\}, \end{aligned} \quad (8.15)$$

where

$$g_1(t) = g_x(t, \bar{x}(t), \bar{u}(t)), \quad g_2(t) = g_u(t, \bar{x}(t), \bar{u}(t)).$$

Now, by the definition of the transposition solution to (1.8) (with y_T and $f(\cdot, \cdot, \cdot)$ given by (8.2)), it follows that

$$\begin{aligned} &-\mathbb{E} \langle h_x(\bar{x}(T)), x_2(T) \rangle_H - \mathbb{E} \int_0^T \langle g_1(t), x_2(t) \rangle_H dt \\ &= \mathbb{E} \int_0^T \left[\langle a_2(t) \delta u(t), y(t) \rangle_H + \langle b_2(t) \delta u(t), Y(t) \rangle_H \right] dt. \end{aligned} \quad (8.16)$$

Combining (8.15) and (8.16), we find

$$\text{Re} \mathbb{E} \int_0^T \langle a_2(t)^* y(t) + b_2(t)^* Y(t) - g_2(t), u(t) - \bar{u}(t) \rangle_{H_1} dt \leq 0 \quad (8.17)$$

holds for any $u(\cdot) \in \mathcal{U}[0, T]$ satisfying $u(\cdot) - \bar{u}(\cdot) \in L_{\mathbb{F}}^2(0, T; L^2(\Omega; H_1))$. Hence, by means of Lemma 2.9, we conclude that

$$\text{Re} \langle a_2(t)^* y(t) + b_2(t)^* Y(t) - g_2(t), u - \bar{u}(t) \rangle_{H_1} \leq 0, \text{ a.e. } [0, T] \times \Omega, \quad \forall u \in U. \quad (8.18)$$

This completes the proof of Theorem 8.1. \square

9 Necessary condition for optimal controls, the case of nonconvex control domains

In this section, we shall derive a necessary condition for optimal controls of the system (1.2) with a general nonconvex control domain. For such a case, the convex perturbation technique does not work any more. We need to adopt the spike variation technique to derive the desired necessary condition.

We need the further following conditions on $a(\cdot, \cdot, \cdot)$, $b(\cdot, \cdot, \cdot)$, $g(\cdot, \cdot, \cdot)$ and $h(\cdot)$:

(A4) The maps $a(t, x, u)$ and $b(t, x, u)$, and the functional $g(t, x, u)$ and $h(x)$ are C^2 with respect to x , and $a_x(t, x, u)$, $b_x(t, x, u)$, $g_x(t, x, u)$, $a_{xx}(t, x, u)$, $b_{xx}(t, x, u)$ and $g_{xx}(t, x, u)$ are continuous

with respect to u . Moreover, there exists a constant $C_L > 0$ such that

$$\left\{ \begin{array}{l} \|a_x(t, x, u)\|_{\mathcal{L}(H)} + \|b_x(t, x, u)\|_{\mathcal{L}(H)} + |g_x(t, x, u)|_H + |h_x(x)|_H \leq C_L, \\ \|a_{xx}(t, x, u)\|_{\mathcal{L}(H \times H, H)} + \|b_{xx}(t, x, u)\|_{\mathcal{L}(H \times H, H)} + \|g_{xx}(t, x, u)\|_{\mathcal{L}(H)} + \|h_{xx}(x)\|_{\mathcal{L}(H)} \leq C_L, \end{array} \right. \quad \forall (t, x, u) \in [0, T] \times H \times U. \quad (9.1)$$

Let

$$\mathbb{H}(t, x, u, k_1, k_2) \triangleq \langle k_1, a(t, x, u) \rangle_H + \langle k_2, b(t, x, u) \rangle_H - g(t, x, u), \quad (9.2)$$

$$(t, x, u, k_1, k_2) \in [0, T] \times H \times U \times H \times H.$$

We have the following result.

Theorem 9.1 Suppose that H is a separable Hilbert space, $L^p_{\mathcal{F}_T}(\Omega; \mathbb{C})$ ($1 \leq p < \infty$) is a separable Banach space, U is a separable metric space, and $x_0 \in L^8_{\mathcal{F}_0}(\Omega; H)$. Let the assumptions (A1), (A2) and (A4) hold, and let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem (P). Let $(y(\cdot), Y(\cdot))$ be the transposition solution to (1.8) with $p = 2$, and y_T and $f(\cdot, \cdot, \cdot)$ given by (8.2). Assume that $b_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))$, and $(P(\cdot), Q(\cdot), \widehat{Q}(\cdot))$ is the relaxed transposition solution to the equation (1.10) in which P_T , $J(\cdot)$, $K(\cdot)$ and $F(\cdot)$ are given by

$$\left\{ \begin{array}{l} P_T = -h_{xx}(\bar{x}(T)), \\ J(t) = a_x(t, \bar{x}(t), \bar{u}(t)), \\ K(t) = b_x(t, \bar{x}(t), \bar{u}(t)), \\ F(t) = -\mathbb{H}_{xx}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t)). \end{array} \right. \quad (9.3)$$

Then,

$$\begin{aligned} & \operatorname{Re} \mathbb{H}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t)) - \operatorname{Re} \mathbb{H}(t, \bar{x}(t), u, y(t), Y(t)) \\ & - \frac{1}{2} \left\langle P(t) \left[b(t, \bar{x}(t), \bar{u}(t)) - b(t, \bar{x}(t), u) \right], b(t, \bar{x}(t), \bar{u}(t)) - b(t, \bar{x}(t), u) \right\rangle_H \geq 0, \end{aligned} \quad (9.4)$$

$$\text{a.e. } [0, T] \times \Omega, \forall u \in U.$$

Proof: We divide the proof into several steps.

Step 1. For each $\varepsilon > 0$, let $E_\varepsilon \subset [0, T]$ be a measurable set with measure ε . Put

$$u^\varepsilon(\cdot) = \begin{cases} \bar{u}(t), & t \in [0, T] \setminus E_\varepsilon, \\ u(t), & t \in E_\varepsilon. \end{cases} \quad (9.5)$$

where $u(\cdot)$ is an arbitrary given element in $\mathcal{U}[0, T]$.

We introduce some notations which will be used in the sequel.

$$\left\{ \begin{array}{l} a_1(t) = a_x(t, \bar{x}(t), \bar{u}(t)), \quad b_1(t) = b_x(t, \bar{x}(t), \bar{u}(t)), \quad g_1(t) = g_x(t, \bar{x}(t), \bar{u}(t)), \\ a_{11}(t) = a_{xx}(t, \bar{x}(t), \bar{u}(t)), \quad b_{11}(t) = b_{xx}(t, \bar{x}(t), \bar{u}(t)), \quad g_{11}(t) = g_{xx}(t, \bar{x}(t), \bar{u}(t)), \\ \tilde{a}_1^\varepsilon(t) = \int_0^1 a_x(t, \bar{x}(t) + \sigma(x^\varepsilon(t) - \bar{x}(t)), u^\varepsilon(t)) d\sigma, \\ \tilde{b}_1^\varepsilon(t) = \int_0^1 b_x(t, \bar{x}(t) + \sigma(x^\varepsilon(t) - \bar{x}(t)), u^\varepsilon(t)) d\sigma, \\ \tilde{a}_{11}^\varepsilon(t) = 2 \int_0^1 (1 - \sigma) a_{xx}(t, \bar{x}(t) + \sigma(x^\varepsilon(t) - \bar{x}(t)), u^\varepsilon(t)) d\sigma, \\ \tilde{b}_{11}^\varepsilon(t) = 2 \int_0^1 (1 - \sigma) b_{xx}(t, \bar{x}(t) + \sigma(x^\varepsilon(t) - \bar{x}(t)), u^\varepsilon(t)) d\sigma, \end{array} \right. \quad (9.6)$$

and

$$\begin{cases} \delta a(t) = a(t, \bar{x}(t), u(t)) - a(t, \bar{x}(t), \bar{u}(t)), \\ \delta b(t) = b(t, \bar{x}(t), u(t)) - b(t, \bar{x}(t), \bar{u}(t)), \\ \delta g(t) = g(t, \bar{x}(t), u(t)) - g(t, \bar{x}(t), \bar{u}(t)), \\ \delta a_1(t) = a_x(t, \bar{x}(t), u(t)) - a_x(t, \bar{x}(t), \bar{u}(t)), \\ \delta b_1(t) = b_x(t, \bar{x}(t), u(t)) - b_x(t, \bar{x}(t), \bar{u}(t)), \\ \delta a_{11}(t) = a_{xx}(t, \bar{x}(t), u(t)) - a_{xx}(t, \bar{x}(t), \bar{u}(t)), \\ \delta b_{11}(t) = b_{xx}(t, \bar{x}(t), u(t)) - b_{xx}(t, \bar{x}(t), \bar{u}(t)), \\ \delta g_1(t) = g_x(t, \bar{x}(t), u(t)) - g_x(t, \bar{x}(t), \bar{u}(t)), \\ \delta g_{11}(t) = g_{xx}(t, \bar{x}(t), u(t)) - g_{xx}(t, \bar{x}(t), \bar{u}(t)). \end{cases} \quad (9.7)$$

Let $x^\varepsilon(\cdot)$ be the state process of the system (1.2) corresponding to the control $u^\varepsilon(\cdot)$. Then, $x^\varepsilon(\cdot)$ solves

$$\begin{cases} dx^\varepsilon = [Ax^\varepsilon + a(t, x^\varepsilon, u^\varepsilon)]dt + b(t, x^\varepsilon, u^\varepsilon)dw(t) & \text{in } (0, T], \\ x^\varepsilon(0) = x_0. \end{cases} \quad (9.8)$$

By Lemma 1.1, we know that

$$|x^\varepsilon|_{C_{\mathbb{F}}([0, T]; L^8(\Omega; H))} \leq C(1 + |x_0|_{L_{\mathcal{F}_0}^8(\Omega; H)}), \quad \forall \varepsilon > 0. \quad (9.9)$$

Let $x_1^\varepsilon(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot)$. Then, by (9.9) and noting that the optimal pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ solves the equation (1.2), we see that $x_1^\varepsilon(\cdot)$ satisfies the following stochastic differential equation:

$$\begin{cases} dx_1^\varepsilon = [Ax_1^\varepsilon + \tilde{a}_1^\varepsilon(t)x_1^\varepsilon + \chi_{E_\varepsilon}(t)\delta a(t)]dt + [\tilde{b}_1^\varepsilon(t)x_1^\varepsilon + \chi_{E_\varepsilon}(t)\delta b(t)]dw(t) & \text{in } (0, T], \\ x_1^\varepsilon(0) = 0. \end{cases} \quad (9.10)$$

Consider the following two stochastic differential equations:

$$\begin{cases} dx_2^\varepsilon = [Ax_2^\varepsilon + a_1(t)x_2^\varepsilon]dt + [b_1(t)x_2^\varepsilon + \chi_{E_\varepsilon}(t)\delta b(t)]dw(t) & \text{in } (0, T], \\ x_2^\varepsilon(0) = 0 \end{cases} \quad (9.11)$$

and²

$$\begin{cases} dx_3^\varepsilon = \left[Ax_3^\varepsilon + a_1(t)x_3^\varepsilon + \chi_{E_\varepsilon}(t)\delta a(t) + \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \right]dt \\ \quad + \left[b_1(t)x_3^\varepsilon + \chi_{E_\varepsilon}(t)\delta b_1(t)x_2^\varepsilon + \frac{1}{2}b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \right]dw(t) & \text{in } (0, T], \\ x_3^\varepsilon(0) = 0. \end{cases} \quad (9.12)$$

In the following Steps 2–4, we shall show that

$$|x_1^\varepsilon - x_2^\varepsilon - x_3^\varepsilon|_{L_{\mathbb{F}}^\infty(0, T; L^2(\Omega; H))} = o(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0. \quad (9.13)$$

Step 2. In this step, we provide some estimates on x_i^ε ($i = 1, 2, 3$).

²Recall that, for any C^2 -function $f(\cdot)$ defined on a Banach space X and $x_0 \in X$, $f_{xx}(x_0) \in \mathcal{L}(X \times X, X)$. This means that, for any $x_1, x_2 \in X$, $f_{xx}(x_0)(x_1, x_2) \in X$. Hence, by (9.6), $a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)$ (in (9.12)) stands for $a_{xx}(t, \bar{x}(t), \bar{u}(t))(x_2^\varepsilon(t), x_2^\varepsilon(t))$. One has a similar meaning for $b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)$ and so on.

First of all, by a direct computation, we find

$$\begin{aligned}
\mathbb{E}|x_1^\varepsilon(t)|_H^8 &= \mathbb{E} \left| \int_0^t S(t-s) \tilde{a}_1^\varepsilon(s) x_1^\varepsilon(s) ds + \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a(s) ds \right. \\
&\quad \left. + \int_0^t S(t-s) \tilde{b}_1^\varepsilon(s) x_1^\varepsilon(s) dw(s) + \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b(s) dw(s) \right|_H^8 \\
&\leq C \left\{ \mathbb{E} \left| \int_0^t S(t-s) \tilde{a}_1^\varepsilon(s) x_1^\varepsilon(s) ds \right|_H^8 + \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a(s) ds \right|_H^8 \right. \\
&\quad \left. + \mathbb{E} \left| \int_0^t S(t-s) \tilde{b}_1^\varepsilon(s) x_1^\varepsilon(s) dw(s) \right|_H^8 + \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b(s) dw(s) \right|_H^8 \right\}. \tag{9.14}
\end{aligned}$$

Now, we estimate the terms in the right hand side of the inequality (9.14) one by one. For the first term, we have

$$\mathbb{E} \left| \int_0^t S(t-s) \tilde{a}_1^\varepsilon(s) x_1^\varepsilon(s) ds \right|_H^8 \leq C \int_0^t \mathbb{E} |\tilde{a}_1^\varepsilon(s) x_1^\varepsilon(s)|_H^8 ds \leq C \mathbb{E} \int_0^t |x_1^\varepsilon(s)|_H^8 ds. \tag{9.15}$$

By the last condition in (1.1), it follows that

$$\begin{aligned}
|\delta a(s)|_H &= |a(s, \bar{x}(s), u(s)) - a(s, \bar{x}(s), \bar{u}(s))|_H \\
&\leq |a(s, \bar{x}(s), u(s)) - a(s, 0, u(s))|_H + |a(s, 0, \bar{u}(s)) - a(s, \bar{x}(s), \bar{u}(s))|_H \\
&\quad + |a(s, 0, u(s)) - a(s, 0, \bar{u}(s))|_H \\
&\leq \left| \int_0^1 a_x(s, \sigma \bar{x}(s), u(s)) \bar{x}(s) d\sigma \right|_H + \left| \int_0^1 a_x(s, \sigma \bar{x}(s), \bar{u}(s)) \bar{x}(s) d\sigma \right|_H + C_L \\
&\leq C[|\bar{x}(s)|_H + 1], \quad \text{a.e. } s \in [0, T]. \tag{9.16}
\end{aligned}$$

Hence, using Lemma 1.1, we have the following estimate:

$$\begin{aligned}
\mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a(s) ds \right|_H^8 &\leq C \mathbb{E} \left\{ \int_0^t \chi_{E_\varepsilon}(s) |\delta a(s)|_H ds \right\}^8 \\
&\leq C \mathbb{E} \left\{ \int_0^T \chi_{E_\varepsilon}(s) [|\bar{x}(s)|_H + 1] ds \right\}^8 \\
&\leq C \mathbb{E} \left\{ \left[\int_0^T \chi_{E_\varepsilon}(s) ds \right]^{7/8} \left[\int_0^T \chi_{E_\varepsilon}(s) (|\bar{x}(s)|_H^8 + 1) ds \right]^{1/8} \right\}^8 \\
&\leq C \varepsilon^7 \int_0^T \chi_{E_\varepsilon}(s) (\mathbb{E} |\bar{x}(s)|_H^8 + 1) ds \leq C(|x(\cdot)|_{C_{\mathbb{F}}([0, T]; L^8(\Omega; H))}^8 + 1) \varepsilon^7 \int_0^T \chi_{E_\varepsilon}(s) ds \\
&\leq C(x_0) \varepsilon^8. \tag{9.17}
\end{aligned}$$

Here and henceforth, $C(x_0)$ is a generic constant (depending on x_0 , T , A and C_L), which may be different from line to line. By Lemma 2.1, similar to (9.15), it follows that

$$\mathbb{E} \left| \int_0^t S(t-s) \tilde{b}_1^\varepsilon(s) x_1^\varepsilon(s) dw(s) \right|_H^8 \leq C \mathbb{E} \left[\int_0^t |S(t-s) \tilde{b}_1^\varepsilon(s) x_1^\varepsilon(s)|_H^2 ds \right]^4 \leq C \mathbb{E} \int_0^t |x_1^\varepsilon(s)|_H^8 ds. \tag{9.18}$$

Similar to (9.16), we have

$$|\delta b(s)|_H \leq C[|\bar{x}(s)|_H + 1], \quad \text{a.e. } s \in [0, T]. \tag{9.19}$$

Hence, by Lemma 2.1 again, similar to (9.17), one has

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b(s) dw(s) \right|_H^8 \leq C \left[\mathbb{E} \int_0^t \chi_{E_\varepsilon}(s) |\delta b(s)|_H^2 ds \right]^4 \\
& \leq C \mathbb{E} \left\{ \int_0^T \chi_{E_\varepsilon}(s) [|\bar{x}(s)|_H^2 + 1] ds \right\}^4 \\
& \leq C \mathbb{E} \left\{ \left[\int_0^T \chi_{E_\varepsilon}(s) ds \right]^{3/4} \left[\int_0^T \chi_{E_\varepsilon}(s) (|\bar{x}(s)|_H^8 + 1) ds \right]^{1/4} \right\}^4 \\
& \leq C \varepsilon^3 \int_0^T \chi_{E_\varepsilon}(s) (\mathbb{E} |\bar{x}(s)|_H^8 + 1) ds \leq C (|x(\cdot)|_{C_{\mathbb{F}}([0,T];L^8(\Omega;H))}^8 + 1) \varepsilon^3 \int_0^T \chi_{E_\varepsilon}(s) ds \\
& \leq C(x_0) \varepsilon^4.
\end{aligned} \tag{9.20}$$

Therefore, combining (9.14), (9.15), (9.17), (9.18) and (9.20), we end up with

$$\mathbb{E} |x_1^\varepsilon(t)|_H^8 \leq C(x_0) \left[\int_0^t \mathbb{E} |x_1^\varepsilon(s)|_H^8 ds + \varepsilon^8 + \varepsilon^4 \right], \quad \text{a.e. } t \in [0, T].$$

This, together with Gronwall's inequality, implies that

$$|x_1^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^8(\Omega;H))}^8 \leq C(x_0) \varepsilon^4. \tag{9.21}$$

From the inequality (9.21) and Hölder's inequality, we find that

$$\begin{aligned}
|x_1^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^4(\Omega;H))}^4 & \leq C(x_0) \varepsilon^2, \\
|x_1^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^2(\Omega;H))}^2 & \leq C(x_0) \varepsilon.
\end{aligned} \tag{9.22}$$

By a similar computation, we have

$$\begin{aligned}
& \mathbb{E} |x_2^\varepsilon(t)|_H^8 \\
& = \mathbb{E} \left| \int_0^t S(t-s) a_1(s) x_2^\varepsilon(s) ds + \int_0^t S(t-s) b_1(s) x_2^\varepsilon(s) dw(s) + \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b(s) dw(s) \right|_H^8 \\
& \leq C \left[\mathbb{E} \left| \int_0^t S(t-s) a_1(s) x_2^\varepsilon(s) ds \right|_H^8 + \mathbb{E} \left| \int_0^t S(t-s) b_1(s) x_2^\varepsilon(s) dw(s) \right|_H^8 \right. \\
& \quad \left. + \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b(s) dw(s) \right|_H^8 \right] \\
& \leq C(x_0) \left(\int_0^t \mathbb{E} |x_2^\varepsilon(s)|_H^8 ds + \varepsilon^4 \right).
\end{aligned} \tag{9.23}$$

By means of Gronwall's inequality once more, (9.23) leads to

$$|x_2^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^8(\Omega;H))}^8 \leq C(x_0) \varepsilon^4. \tag{9.24}$$

From inequality (9.24) and utilizing Hölder's inequality again, we get

$$\begin{aligned}
|x_2^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^4(\Omega;H))}^4 & \leq C(x_0) \varepsilon^2, \\
|x_2^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^2(\Omega;H))}^2 & \leq C(x_0) \varepsilon.
\end{aligned} \tag{9.25}$$

Similar to (9.17), we have

$$\mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a(s) ds \right|_H^4 \leq C(x_0) \varepsilon^4.$$

Hence, it follows from Lemma 2.1 and (9.24)–(9.25) that

$$\begin{aligned}
& |x_3^\varepsilon(t)|_{L^4_{\mathcal{F}_t}(\Omega;H)}^4 \\
&= \mathbb{E} \left| \int_0^t S(t-s)a_1(s)x_3^\varepsilon(s)ds + \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\delta a(s)ds + \frac{1}{2} \int_0^t S(t-s)a_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s))ds \right. \\
&\quad + \int_0^t S(t-s)b_1(s)x_3^\varepsilon(s)dw(s) + \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\delta b_1(s)x_2^\varepsilon(s)dw(s) \\
&\quad \left. + \frac{1}{2} \int_0^t S(t-s)b_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s))dw(s) \right|_H^4 \\
&\leq C(x_0)\mathbb{E} \left[\int_0^t |x_3^\varepsilon(s)|_H^4 ds + \left| \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\delta a(s)ds \right|_H^4 + \int_0^T |x_2^\varepsilon(t)|_H^8 ds + \left| \int_0^T |\chi_{E_\varepsilon}x_2^\varepsilon(t)|_H^2 dt \right|^2 \right] \\
&\leq C(x_0) \left[\mathbb{E} \int_0^t |x_3^\varepsilon(s)|_H^4 ds + \varepsilon^4 + \mathbb{E} \left(\left| \int_0^T \chi_{E_\varepsilon}(s)ds \right|^{1/2} \left| \int_0^T \chi_{E_\varepsilon}(s)|x_2^\varepsilon(s)|_H^4 ds \right|^{1/2} \right)^2 \right] \\
&\leq C(x_0) \left[\mathbb{E} \int_0^t |x_3^\varepsilon(s)|_H^4 ds + \varepsilon^4 + \varepsilon \int_0^T \chi_{E_\varepsilon}(s)\mathbb{E}|x_2^\varepsilon(s)|_H^4 ds \right] \\
&\leq C(x_0) \left[\mathbb{E} \int_0^t |x_3^\varepsilon(s)|_H^4 ds + \varepsilon^4 \right],
\end{aligned}$$

this, together with Gronwall's inequality, implies that

$$|x_3^\varepsilon(\cdot)|_{L^\infty_{\mathbb{F}}(0,T;L^4(\Omega;H))}^4 \leq C(x_0)\varepsilon^4. \quad (9.26)$$

Then, by Hölder's inequality, we conclude that

$$|x_3^\varepsilon(\cdot)|_{L^\infty_{\mathbb{F}}(0,T;L^2(\Omega;H))}^2 \leq C(x_0)\varepsilon^2. \quad (9.27)$$

Step 3. We now estimate $x_4^\varepsilon \triangleq x_1^\varepsilon - x_2^\varepsilon$. Clearly, x_4^ε solves the following equation:

$$\begin{cases} dx_4^\varepsilon = \{Ax_4^\varepsilon + \tilde{a}_1^\varepsilon(t)x_4^\varepsilon + [\tilde{a}_1^\varepsilon(t) - a_1(t)]x_2^\varepsilon + \chi_{E_\varepsilon}(t)\delta a(t)\}dt \\ \quad + \{\tilde{b}_1^\varepsilon(t)x_4^\varepsilon + [\tilde{b}_1^\varepsilon(t) - b_1(t)]x_2^\varepsilon\}dw(t) & \text{in } (0, T], \\ x_4^\varepsilon(0) = 0. \end{cases} \quad (9.28)$$

Hence,

$$\begin{aligned}
& \mathbb{E}|x_4^\varepsilon(t)|_H^2 \\
&= \mathbb{E} \left| \int_0^t S(t-s)\tilde{a}_1^\varepsilon(s)x_4^\varepsilon(s)ds + \int_0^t S(t-s)[\tilde{a}_1^\varepsilon(s) - a_1(s)]x_2^\varepsilon(s)ds + \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\delta a(s)ds \right. \\
&\quad + \int_0^t S(t-s)\tilde{b}_1^\varepsilon(s)x_4^\varepsilon(s)dw(s) + \int_0^t S(t-s)[\tilde{b}_1^\varepsilon(s) - b_1(s)]x_2^\varepsilon(s)dw(s) \Big|_H^2 \\
&\leq C \left[\mathbb{E} \left| \int_0^t S(t-s)\tilde{a}_1^\varepsilon(s)x_4^\varepsilon(s)ds \right|_H^2 + \mathbb{E} \left| \int_0^t S(t-s)[\tilde{a}_1^\varepsilon(s) - a_1(s)]x_2^\varepsilon(s)ds \right|_H^2 \right. \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\delta a(s)ds \right|_H^2 + \mathbb{E} \left| \int_0^t S(t-s)\tilde{b}_1^\varepsilon(s)x_4^\varepsilon(s)dw(s) \right|_H^2 \\
&\quad \left. + \mathbb{E} \left| \int_0^t S(t-s)[\tilde{b}_1^\varepsilon(s) - b_1(s)]x_2^\varepsilon(s)dw(s) \right|_H^2 \right].
\end{aligned} \quad (9.29)$$

We now estimate the terms in the right hand side of the inequality (9.29) one by one. It is easy to see that

$$\mathbb{E} \left| \int_0^t S(t-s) \tilde{a}_1^\varepsilon(s) x_4^\varepsilon(s) ds \right|_H^2 \leq C \mathbb{E} \int_0^t |\tilde{a}_1^\varepsilon(s) x_4^\varepsilon(s)|_H^2 ds \leq C(x_0) \mathbb{E} \int_0^t |x_4^\varepsilon(s)|^2 ds. \quad (9.30)$$

By (9.25), we have

$$\begin{aligned} & \|\tilde{a}_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)} \\ &= \left\| \int_0^1 [a_x(s, \bar{x}(s) + \sigma x_1^\varepsilon(w), u^\varepsilon(s)) - a_x(s, \bar{x}(s), \bar{u}(s))] d\sigma \right\|_{\mathcal{L}(H)} \\ &= \left\| \int_0^1 [a_x(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), u^\varepsilon(s)) - a_x(s, \bar{x}(s), u^\varepsilon(s)) \right. \\ & \quad \left. + a_x(s, \bar{x}(s), u^\varepsilon(s)) - a_x(s, \bar{x}(s), \bar{u}(s))] d\sigma \right\|_{\mathcal{L}(H)} \\ &= \left\| \int_0^1 \left[\sigma \int_0^1 a_{xx}(s, \bar{x}(s) + \eta \sigma x_1^\varepsilon(s), u^\varepsilon(s)) x_1^\varepsilon(s) d\eta + \chi_{E_\varepsilon}(s) \delta a_1(s) \right] d\sigma \right\|_{\mathcal{L}(H)} \\ &\leq C \left[|x_1^\varepsilon(s)|_H + \chi_{E_\varepsilon}(s) \right], \quad \text{a.e. } s \in [0, T]. \end{aligned} \quad (9.31)$$

Hence,

$$\begin{aligned} & \mathbb{E} \left| \int_0^t S(t-s) [\tilde{a}_1^\varepsilon(s) - a_1(s)] x_2^\varepsilon(s) ds \right|_H^2 \\ &\leq C \mathbb{E} \int_0^t \|\tilde{a}_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)}^2 |x_2^\varepsilon(s)|_H^2 ds \\ &\leq C |x_2^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0, T; L^4(\Omega; H))}^2 \int_0^T \left[\mathbb{E} \|\tilde{a}_1^\varepsilon(s) - a_1(s)\|_{\mathcal{L}(H)}^4 \right]^{1/2} ds \\ &\leq C(x_0) \varepsilon \int_0^T \left[\chi_{E_\varepsilon}(t) + \mathbb{E} |x_1^\varepsilon(t)|_H^4 \right]^{1/2} dt \\ &\leq C(x_0) \varepsilon \int_0^T \left[\chi_{E_\varepsilon}(t) + \left(\mathbb{E} |x_1^\varepsilon(t)|_H^4 \right)^{1/2} \right] dt \leq C(x_0) \varepsilon^2. \end{aligned} \quad (9.32)$$

Similar to (9.17), we have

$$\mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a(s) ds \right|_H^2 \leq C(x_0) \varepsilon^2. \quad (9.33)$$

By Lemma 2.1 and similar to (9.30), it follows that

$$\begin{aligned} & \mathbb{E} \left| \int_0^t S(t-s) \tilde{b}_1^\varepsilon(s) x_4^\varepsilon(s) dw(s) \right|_H^2 \leq C \mathbb{E} \int_0^t |S(t-s) \tilde{b}_1^\varepsilon(s) x_4^\varepsilon(s)|_H^2 ds \\ &\leq C \mathbb{E} \int_0^t |\tilde{b}_1^\varepsilon(s) x_4^\varepsilon(s)|^2 ds \leq C \mathbb{E} \int_0^t |x_4^\varepsilon(s)|^2 ds. \end{aligned} \quad (9.34)$$

Similar to (9.31), we have

$$\|\tilde{b}_1^\varepsilon(s) - b_1(s)\|_{\mathcal{L}(H)} \leq C \left[|x_1^\varepsilon(s)|_H + \chi_{E_\varepsilon}(s) \right], \quad \text{a.e. } s \in [0, T].$$

Hence, similar to (9.32), one obtains that

$$\mathbb{E} \left| \int_0^t S(t-s) [\tilde{b}_1^\varepsilon(s) - b_1(s)] x_2^\varepsilon(s) dw(s) \right|_H^2 \leq C \mathbb{E} \int_0^t \|\tilde{b}_1^\varepsilon(s) - b_1(s)\|_{\mathcal{L}(H)}^2 |x_2^\varepsilon(s)|_H^2 ds \leq C(x_0) \varepsilon^2. \quad (9.35)$$

Combining (9.29), (9.30), (9.32), (9.33), (9.34) and (9.35), we obtain that

$$\mathbb{E}|x_4^\varepsilon(t)|_H^2 \leq C(x_0) \left(\int_0^t \mathbb{E}|x_4^\varepsilon(s)|_H^2 ds + \varepsilon^2 \right).$$

Utilizing Gronwall's inequality again, we find that

$$|x_4^\varepsilon(\cdot)|_{L^\infty_{\mathbb{F}}(0,T;L^2(\Omega;H))}^2 \leq C(x_0)\varepsilon^2, \quad \forall t \in [0, T]. \quad (9.36)$$

Step 4. We are now in a position to estimate $\mathbb{E}|x_1^\varepsilon(t) - x_2^\varepsilon(t) - x_3^\varepsilon(t)|_H^2 = \mathbb{E}|x_4^\varepsilon(t) - x_3^\varepsilon(t)|_H^2$.

Let $x_5^\varepsilon(\cdot) = x_4^\varepsilon(\cdot) - x_3^\varepsilon(\cdot)$. It is clear that $x_5^\varepsilon(\cdot) = x_1^\varepsilon(t) - x_2^\varepsilon(t) - x_3^\varepsilon(t) = x^\varepsilon(\cdot) - \bar{x}(\cdot) - x_2^\varepsilon(t) - x_3^\varepsilon(t)$.

We claim that $x_5^\varepsilon(\cdot)$ solves the following equation (Recall (9.6)–(9.7) for the notations):

$$\left\{ \begin{array}{l} dx_5^\varepsilon = \left[Ax_5^\varepsilon + a_1(t)x_5^\varepsilon + \chi_{E_\varepsilon}(t)\delta a_1(t)x_1^\varepsilon + \frac{1}{2}(\tilde{a}_{11}^\varepsilon(t) - a_{xx}(t, \bar{x}, u^\varepsilon))(x_1^\varepsilon, x_1^\varepsilon) \right. \\ \quad \left. + \frac{1}{2}\chi_{E_\varepsilon}(t)\delta a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \right] dt \\ \quad + \left[b_1(t)x_5^\varepsilon + \chi_{E_\varepsilon}(t)\delta b_1(t)x_4^\varepsilon + \frac{1}{2}(\tilde{b}_{11}^\varepsilon(t) - b_{xx}(t, \bar{x}, u^\varepsilon))(x_1^\varepsilon, x_1^\varepsilon) \right. \\ \quad \left. + \frac{1}{2}\chi_{E_\varepsilon}(t)\delta b_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}b_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \right] dw(t), \quad \text{in } (0, T], \\ x_5^\varepsilon(0) = 0. \end{array} \right. \quad (9.37)$$

Indeed, by (9.8), (1.2), (9.11) and (9.12), it is easy to see that the drift term for the equation solved by $x_5^\varepsilon(\cdot)$ is as follows:

$$\begin{aligned} & Ax^\varepsilon + a(t, x^\varepsilon, u^\varepsilon) - A\bar{x} - a(t, \bar{x}, \bar{u}) - Ax_2^\varepsilon - a_1(t)x_2^\varepsilon - Ax_3^\varepsilon - a_1(t)x_3^\varepsilon \\ & \quad - \chi_{E_\varepsilon}(t)\delta a(t) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \\ & = Ax_5^\varepsilon + a(t, x^\varepsilon, u^\varepsilon) - a(t, \bar{x}, u^\varepsilon) - a_1(t)(x_2^\varepsilon + x_3^\varepsilon) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon). \end{aligned} \quad (9.38)$$

For $\sigma \in [0, 1]$, write $f(\sigma) = a(t, \bar{x} + \sigma x_1^\varepsilon, u^\varepsilon)$. Then, by Taylor's formula with the integral type remainder, we see that

$$f(1) - f(0) = f'(0) + \int_0^1 (1 - \sigma)f''(\sigma)d\sigma.$$

Since $f'(\sigma) = a_x(t, \bar{x} + \sigma x_1^\varepsilon, u^\varepsilon)x_1^\varepsilon$ and $f''(\sigma) = a_{xx}(t, \bar{x} + \sigma x_1^\varepsilon, u^\varepsilon)(x_1^\varepsilon, x_1^\varepsilon)$, we obtain that

$$\begin{aligned} a(t, x^\varepsilon, u^\varepsilon) - a(t, \bar{x}, u^\varepsilon) &= a_x(t, \bar{x}, u^\varepsilon)x_1^\varepsilon + \int_0^1 (1 - \sigma)a_{xx}(t, \bar{x} + \sigma x_1^\varepsilon, u^\varepsilon)(x_1^\varepsilon, x_1^\varepsilon)d\sigma \\ &= a_x(t, \bar{x}, u^\varepsilon)x_1^\varepsilon + \frac{1}{2}\tilde{a}_{11}^\varepsilon(t)(x_1^\varepsilon, x_1^\varepsilon). \end{aligned} \quad (9.39)$$

Next,

$$\begin{aligned} & a_x(t, \bar{x}, u^\varepsilon)x_1^\varepsilon - a_1(t)(x_2^\varepsilon + x_3^\varepsilon) \\ &= a_x(t, \bar{x}, u^\varepsilon)x_1^\varepsilon - a_x(t, \bar{x}, \bar{u})x_1^\varepsilon + a_1(t)(x_1^\varepsilon - x_2^\varepsilon - x_3^\varepsilon) \\ &= \chi_{E_\varepsilon}[a_x(t, \bar{x}, u) - a_x(t, \bar{x}, \bar{u})]x_1^\varepsilon + a_1(t)x_5^\varepsilon \\ &= \chi_{E_\varepsilon}\delta a_1(t)x_1^\varepsilon + a_1(t)x_5^\varepsilon. \end{aligned} \quad (9.40)$$

Further,

$$\begin{aligned}
& \frac{1}{2}\tilde{a}_{11}^\varepsilon(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \\
&= \frac{1}{2}\tilde{a}_{11}^\varepsilon(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{xx}(t, \bar{x}, u^\varepsilon)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}a_{xx}(t, \bar{x}, u^\varepsilon)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) \\
&\quad + \frac{1}{2}a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \\
&= \frac{1}{2}\left(\tilde{a}_{11}^\varepsilon(t) - a_{xx}(t, \bar{x}, u^\varepsilon)\right)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}\chi_{E_\varepsilon}\delta a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon).
\end{aligned} \tag{9.41}$$

By (9.38)–(9.41), we conclude that

$$\begin{aligned}
& Ax^\varepsilon + a(t, x^\varepsilon, u^\varepsilon) - A\bar{x} - a(t, \bar{x}, \bar{u}) - Ax_2^\varepsilon - a_1(t)x_2^\varepsilon - Ax_3^\varepsilon - a_1(t)x_3^\varepsilon \\
& - \chi_{E_\varepsilon}(t)\delta a(t) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) - \chi_{E_\varepsilon}(t)\delta b_1(t)x_2^\varepsilon \\
&= Ax_5^\varepsilon + a_1(t)x_5^\varepsilon + \chi_{E_\varepsilon}(t)\delta a_1(t)x_1^\varepsilon + \frac{1}{2}(\tilde{a}_{11}^\varepsilon(t) - a_{xx}(t, \bar{x}, u^\varepsilon))(x_1^\varepsilon, x_1^\varepsilon) \\
& \quad + \frac{1}{2}\chi_{E_\varepsilon}(t)\delta a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}a_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon).
\end{aligned}$$

Similarly, the diffusion term (for the equation solved by $x_5^\varepsilon(\cdot)$) is as follows:

$$\begin{aligned}
& b(t, x^\varepsilon, u^\varepsilon) - b(t, \bar{x}, \bar{u}) - b_1(t)x_2^\varepsilon - b_1(t)x_3^\varepsilon - \chi_{E_\varepsilon}(t)\delta b(t)x_2^\varepsilon - \frac{1}{2}b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon) \\
&= b_1(t)x_5^\varepsilon + \chi_{E_\varepsilon}(t)\delta b_1(t)x_4^\varepsilon + \frac{1}{2}(\tilde{b}_{11}^\varepsilon(t) - b_{xx}(t, \bar{x}, u^\varepsilon))(x_1^\varepsilon, x_1^\varepsilon) \\
& \quad + \frac{1}{2}\chi_{E_\varepsilon}(t)\delta b_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) + \frac{1}{2}b_{11}(t)(x_1^\varepsilon, x_1^\varepsilon) - \frac{1}{2}b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon).
\end{aligned}$$

This verifies that $x_5^\varepsilon(\cdot)$ satisfies the equation (9.37).

From (9.37), we see that, for any $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}|x_5^\varepsilon(t)|_H^2 &\leq C \left[\mathbb{E} \left| \int_0^t S(t-s) [a_1(s)x_5^\varepsilon(s) + \chi_{E_\varepsilon}(s)\delta a_1(s)x_1^\varepsilon(s)] ds \right|_H^2 \right. \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s) (\tilde{a}_{11}^\varepsilon(s) - a_{xx}(s, \bar{x}(s), u^\varepsilon(s)))(x_1^\varepsilon(s), x_1^\varepsilon(s)) ds \right|_H^2 \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) ds \right|_H^2 \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s) [a_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) - a_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s))] ds \right|_H^2 \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s) [b_1(s)x_5^\varepsilon(s) + \chi_{E_\varepsilon}(s)\delta b_1(s)x_4^\varepsilon(s)] dw(s) \right|_H^2 \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s) (\tilde{b}_{11}^\varepsilon(s) - b_{xx}(s, \bar{x}(s), u^\varepsilon(s)))(x_1^\varepsilon(s), x_1^\varepsilon(s)) dw(s) \right|_H^2 \\
&\quad + \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) dw(s) \right|_H^2 \\
&\quad \left. + \mathbb{E} \left| \int_0^t S(t-s) [b_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) - b_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s))] dw(s) \right|_H^2 \right].
\end{aligned} \tag{9.42}$$

We now estimate the “drift” terms in the right hand side of (9.42). By (9.22), we have the following estimate:

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) [a_1(s)x_5^\varepsilon(s) + \chi_{E_\varepsilon}(s)\delta a_1(s)x_1^\varepsilon(s)] ds \right|_H^2 \\
& \leq C \left[\mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + \mathbb{E} \left| \int_0^T \chi_{E_\varepsilon}(s) |x_1^\varepsilon(\cdot)|_H ds \right|^2 \right] \\
& \leq C \left[\mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + \mathbb{E} \left| \int_0^T \chi_{E_\varepsilon}(s) ds \int_0^T \chi_{E_\varepsilon}(s) |x_1^\varepsilon(s)|_H^2 ds \right| \right] \\
& \leq C \left[\mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + \varepsilon \int_0^T \chi_{E_\varepsilon}(s) \mathbb{E} |x_1^\varepsilon(s)|_H^2 ds \right] \\
& \leq C \left[\mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + \varepsilon |x_1^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^2(\Omega;H))}^2 \int_0^t \chi_{E_\varepsilon}(s) ds \right] \\
& \leq C(x_0) \left[\mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + \varepsilon^3 \right].
\end{aligned} \tag{9.43}$$

By (9.6) and recalling that $x_1^\varepsilon(\cdot) = x^\varepsilon(\cdot) - \bar{x}(\cdot)$, we see that, for a.e. $s \in [0, T]$,

$$\begin{aligned}
& \left\| \tilde{a}_{11}^\varepsilon(s) - a_{xx}(s, \bar{x}(s), u^\varepsilon(s)) \right\|_{\mathcal{L}(H \times H, H)} \\
& = \left\| 2 \int_0^1 (1-\sigma) a_{xx}(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), u^\varepsilon(s)) d\sigma - a_{xx}(s, \bar{x}(s), u^\varepsilon(s)) \right\|_{\mathcal{L}(H \times H, H)} \\
& = \left\| 2 \int_0^1 (1-\sigma) \left[a_{xx}(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), \bar{u}(s)) - a_{xx}(s, \bar{x}(s), \bar{u}(s)) \right] d\sigma \right. \\
& \quad \left. + 2 \int_0^1 (1-\sigma) \chi_{E_\varepsilon}(s) a_{xx}(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), u(s)) d\sigma + \chi_{E_\varepsilon}(s) a_{xx}(s, \bar{x}(s), u(s)) \right\|_{\mathcal{L}(H \times H, H)} \\
& \leq C \left[\int_0^1 \left\| a_{xx}(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), \bar{u}(s)) - a_{xx}(s, \bar{x}(s), \bar{u}(s)) \right\|_{\mathcal{L}(H \times H, H)} d\sigma + \chi_{E_\varepsilon}(s) \right].
\end{aligned} \tag{9.44}$$

Hence, by (9.21) and noting the continuity of $a_{xx}(t, x, u)$ with respect to x , we have

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) (\tilde{a}_{11}^\varepsilon(s) - a_{xx}(s, \bar{x}(s), u^\varepsilon(s))) (x_1^\varepsilon(s), x_1^\varepsilon(s)) ds \right|_H^2 \\
& \leq C \mathbb{E} \int_0^T \left\| \tilde{a}_{11}^\varepsilon(s) - a_{xx}(s, \bar{x}(s), u^\varepsilon(s)) \right\|_{\mathcal{L}(H \times H, H)}^2 |x_1^\varepsilon(s)|_H^4 dt \\
& \leq C |x_1^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^8(\Omega;H))}^4 \int_0^T \left[\mathbb{E} \left\| \tilde{a}_{11}^\varepsilon(s) - a_{xx}(s, \bar{x}(s), u^\varepsilon(s)) \right\|_{\mathcal{L}(H \times H, H)}^4 \right]^{1/2} ds \\
& \leq C(x_0) \varepsilon^2 \int_0^T \left[\mathbb{E} \int_0^1 \left\| a_{xx}(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), \bar{u}(s)) - a_{xx}(s, \bar{x}(s), \bar{u}(s)) \right\|_{\mathcal{L}(H \times H, H)}^4 d\sigma + \chi_{E_\varepsilon}(s) \right]^{1/2} ds \\
& = o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned} \tag{9.45}$$

Also, it holds that

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta a_{11}(s) (x_1^\varepsilon(s), x_1^\varepsilon(s)) ds \right|_H^2 \\
& \leq C |x_1^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0,T;L^8(\Omega;H))}^4 \int_0^T \chi_{E_\varepsilon}(s) \left[\mathbb{E} \left\| \delta a_{11}(s) \right\|_{\mathcal{L}(H \times H, H)}^4 \right]^{1/2} ds \\
& \leq C(x_0) \varepsilon^2 \int_0^T \chi_{E_\varepsilon}(t) dt = C(x_0) \varepsilon^3.
\end{aligned} \tag{9.46}$$

By means of (9.22), (9.25) and (9.36), and noting that $x_4^\varepsilon = x_1^\varepsilon - x_2^\varepsilon$, we obtain that

$$\begin{aligned}
& \frac{1}{2} \mathbb{E} \left| \int_0^t S(t-s) \left[a_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) - a_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s)) \right] ds \right|_H^2 \\
&= \frac{1}{2} \mathbb{E} \left| \int_0^t S(t-s) \left[a_{11}(s)(x_4^\varepsilon(s), x_1^\varepsilon(s)) + a_{11}(s)(x_2^\varepsilon(s), x_4^\varepsilon(s)) \right] ds \right|_H^2 \\
&\leq C \left(|x_1^\varepsilon(\cdot)|_{L^\infty(0,T;L^2(\Omega;H))}^2 + |x_2^\varepsilon(\cdot)|_{L^\infty(0,T;L^2(\Omega;H))}^2 |x_4^\varepsilon(\cdot)|_{L^\infty(0,T;L^2(\Omega;H))}^2 \right) \\
&\leq C(x_0)\varepsilon^3.
\end{aligned} \tag{9.47}$$

Next, we estimate the “diffusion” terms in the right hand side of (9.42). Similar to (9.43) and noting (9.36), we obtain that

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) [b_1(s)x_5^\varepsilon(s) + \chi_{E_\varepsilon}(s)\delta b_1(s)x_4^\varepsilon(s)] dw(s) \right|_H^2 \\
&\leq \mathbb{E} \int_0^t |S(t-s)[b_1(s)x_5^\varepsilon(s) + \chi_{E_\varepsilon}(s)\delta b_1(s)x_4^\varepsilon(s)]|_H^2 ds \leq C(x_0) \left[\mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + \varepsilon^3 \right].
\end{aligned} \tag{9.48}$$

By virtue of (9.21) again, similar to (9.45), we find that

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) (\tilde{b}_{11}^\varepsilon(s) - b_{xx}(s, \bar{x}(s), u^\varepsilon(s))) (x_1^\varepsilon(s), x_1^\varepsilon(s)) dw(s) \right|_H^2 \\
&= \mathbb{E} \int_0^t \left| S(t-s) (\tilde{b}_{11}^\varepsilon(s) - b_{xx}(s, \bar{x}(s), u^\varepsilon(s))) (x_1^\varepsilon(s), x_1^\varepsilon(s)) \right|_H^2 ds \\
&\leq C |x_1^\varepsilon(\cdot)|_{L^\infty(0,T;L^8(\Omega;H))}^4 \int_0^T \left[\mathbb{E} \|\tilde{b}_{11}^\varepsilon(s) - b_{xx}(s, \bar{x}(s), u^\varepsilon(s))\|_{\mathcal{L}(H \times H, H)}^4 \right]^{1/2} ds \\
&\leq C(x_0)\varepsilon^2 \int_0^T \left[\mathbb{E} \int_0^1 \|b_{xx}(s, \bar{x}(s) + \sigma x_1^\varepsilon(s), \bar{u}(s)) - b_{xx}(s, \bar{x}(s), \bar{u}(s))\|_{\mathcal{L}(H \times H, H)}^4 d\sigma + \chi_{E_\varepsilon}(s) \right]^{1/2} ds \\
&= o(\varepsilon^2), \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned} \tag{9.49}$$

Similar to (9.46), we have

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) \chi_{E_\varepsilon}(s) \delta b_{11}(s) (x_1^\varepsilon(s), x_1^\varepsilon(s)) dw(s) \right|_H^2 \\
&= \mathbb{E} \int_0^t |S(t-s) \chi_{E_\varepsilon}(s) \delta b_{11}(s) (x_1^\varepsilon(s), x_1^\varepsilon(s))|_H^2 ds \leq C(x_0)\varepsilon^3.
\end{aligned} \tag{9.50}$$

Similar to (9.47), it holds that

$$\begin{aligned}
& \mathbb{E} \left| \int_0^t S(t-s) [b_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) - b_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s))] dw(s) \right|_H^2 \\
&= \mathbb{E} \int_0^t \left| S(t-s) [b_{11}(s)(x_1^\varepsilon(s), x_1^\varepsilon(s)) - b_{11}(s)(x_2^\varepsilon(s), x_2^\varepsilon(s))] \right|_H^2 ds \leq C(x_0)\varepsilon^3.
\end{aligned} \tag{9.51}$$

From (9.42)–(9.43) and (9.45)–(9.51), we conclude that

$$\mathbb{E} |x_5^\varepsilon(t)|_H^2 \leq C(x_0) \mathbb{E} \int_0^t |x_5^\varepsilon(s)|_H^2 ds + o(\varepsilon^2), \quad \text{as } t \rightarrow 0. \tag{9.52}$$

By means of Gronwall’s inequality again, we get

$$|x_5^\varepsilon(\cdot)|_{L^\infty(0,T;L^2(\Omega;H))}^2 = o(\varepsilon^2), \quad \text{as } t \rightarrow 0. \tag{9.53}$$

This gives (9.13).

Step 5. We are now in a position to complete the proof.

We need to compute the value of $\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot))$.

$$\begin{aligned}
& \mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\
&= \mathbb{E} \int_0^T [g(t, x^\varepsilon(t), u^\varepsilon(t)) - g(t, \bar{x}(t), \bar{u}(t))] dt + \mathbb{E}h(x^\varepsilon(T)) - \mathbb{E}h(\bar{x}(T)) \\
&= \operatorname{Re} \mathbb{E} \int_0^T \left\{ \chi_{E_\varepsilon}(t) \delta g(t) + \langle g_x(t, \bar{x}(t), u^\varepsilon(t)), x_1^\varepsilon(t) \rangle_H \right. \\
&\quad \left. + \int_0^1 \langle (1-\sigma)g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) x_1^\varepsilon(t), x_1^\varepsilon(t) \rangle_H d\sigma \right\} dt \\
&\quad + \operatorname{Re} \mathbb{E} \langle h_x(\bar{x}(T)), x_1^\varepsilon(T) \rangle_H + \operatorname{Re} \mathbb{E} \int_0^1 \langle (1-\sigma)h_{xx}(\bar{x}(T) + \sigma x_1^\varepsilon(T)) x_1^\varepsilon(T), x_1^\varepsilon(T) \rangle_H d\sigma.
\end{aligned} \tag{9.54}$$

This, together with the definition of $x_1^\varepsilon(\cdot)$, $x_2^\varepsilon(\cdot)$, $x_3^\varepsilon(\cdot)$, $x_4^\varepsilon(\cdot)$ and $x_5^\varepsilon(\cdot)$, yields that

$$\begin{aligned}
& \mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\
&= \operatorname{Re} \mathbb{E} \int_0^T \left\{ \chi_{E_\varepsilon}(t) \delta g(t) + \langle \delta g_1(t), x_1^\varepsilon(t) \rangle_H \chi_{E_\varepsilon}(t) + \langle g_1(t), x_2^\varepsilon(t) + x_3^\varepsilon(t) \rangle_H + \langle g_1(t), x_5^\varepsilon(t) \rangle_H \right. \\
&\quad \left. + \int_0^1 \langle (1-\sigma)[g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) - g_{xx}(t, \bar{x}(t), u^\varepsilon(t))] x_1^\varepsilon(t), x_1^\varepsilon(t) \rangle_H d\sigma \right. \\
&\quad \left. + \frac{1}{2} \langle \delta g_{11}(t) x_1^\varepsilon(t), x_1^\varepsilon(t) \rangle_H \chi_{E_\varepsilon}(t) + \frac{1}{2} \langle g_{11}(t) x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H + \frac{1}{2} \langle g_{11}(t) x_4^\varepsilon(t), x_1^\varepsilon(t) + x_2^\varepsilon(t) \rangle_H \right\} dt \\
&\quad + \operatorname{Re} \mathbb{E} \langle h_x(\bar{x}(T)), x_2^\varepsilon(T) + x_3^\varepsilon(T) \rangle_H + \operatorname{Re} \mathbb{E} \langle h_x(\bar{x}(T)), x_5^\varepsilon(T) \rangle_H + \frac{1}{2} \operatorname{Re} \mathbb{E} \langle h_{xx}(\bar{x}(T)) x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H \\
&\quad + \frac{1}{2} \operatorname{Re} \mathbb{E} \langle h_{xx}(\bar{x}(T)) x_4^\varepsilon(T), x_1^\varepsilon(T) + x_2^\varepsilon(T) \rangle_H \\
&\quad + \operatorname{Re} \mathbb{E} \int_0^1 \langle (1-\sigma)[h_{xx}(\bar{x}(T) + \sigma x_1^\varepsilon(T)) - h_{xx}(\bar{x}(T))] x_1^\varepsilon(T), x_1^\varepsilon(T) \rangle_H d\sigma.
\end{aligned} \tag{9.55}$$

Similar to (9.44), for a.e. $t \in [0, T]$, we find that

$$\begin{aligned}
& \left\| \int_0^1 (1-\sigma)[g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u^\varepsilon(t)) - g_{xx}(t, \bar{x}(t), u^\varepsilon(t))] d\sigma \right\|_{\mathcal{L}(H \times H, H)} \\
&= \left\| \int_0^1 (1-\sigma)[g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), \bar{u}(t)) - g_{xx}(t, \bar{x}(t), \bar{u}(t))] d\sigma \right. \\
&\quad \left. + \int_0^1 (1-\sigma)\chi_{E_\varepsilon}(t)g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), u(t)) d\sigma + \chi_{E_\varepsilon}(t)g_{xx}(t, \bar{x}(t), u(t)) \right\|_{\mathcal{L}(H \times H, H)} d\sigma \\
&\leq C \left[\int_0^1 \|g_{xx}(t, \bar{x}(t) + \sigma x_1^\varepsilon(t), \bar{u}(t)) - g_{xx}(t, \bar{x}(t), \bar{u}(t))\|_{\mathcal{L}(H \times H, H)} d\sigma + \chi_{E_\varepsilon}(t) \right].
\end{aligned} \tag{9.56}$$

By (9.55), noting (9.21), (9.24), (9.26), (9.36), (9.53) and (9.56), and using the continuity of both $h_{xx}(x)$ and $g_{xx}(x)$ with respect to x , we end up with

$$\begin{aligned}
& \mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\
&= \operatorname{Re} \mathbb{E} \int_0^T \left[\langle g_1(t), x_2^\varepsilon(t) + x_3^\varepsilon(t) \rangle_H + \frac{1}{2} \langle g_{11}(t) x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H + \chi_{E_\varepsilon}(t) \delta g(t) \right] dt \\
&\quad + \operatorname{Re} \mathbb{E} \langle h_x(\bar{x}(T)), x_2^\varepsilon(T) + x_3^\varepsilon(T) \rangle_H + \frac{1}{2} \operatorname{Re} \mathbb{E} \langle h_{xx}(\bar{x}(T)) x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H + o(\varepsilon).
\end{aligned} \tag{9.57}$$

In the sequel, we shall get rid of $x_2^\varepsilon(\cdot)$ and $x_3^\varepsilon(\cdot)$ in (9.57) by solutions to the equations (1.8) and (1.10).

First, by the definition of the transposition solution to (1.8) (with y_T and $f(\cdot, \cdot, \cdot)$ given by (8.2)), we obtain that

$$-\mathbb{E}\langle h_x(\bar{x}_T), x_2^\varepsilon(T) \rangle_H - \mathbb{E} \int_0^T \langle g_1(t), x_2^\varepsilon(t) \rangle_H dt = \mathbb{E} \int_0^T \langle Y(t), \delta b(t) \rangle_H \chi_{E_\varepsilon}(t) dt \quad (9.58)$$

and

$$\begin{aligned} & -\mathbb{E}\langle h_x(\bar{x}_T), x_3^\varepsilon(T) \rangle_H - \mathbb{E} \int_0^T \langle g_1(t), x_3^\varepsilon(t) \rangle_H dt \\ &= \mathbb{E} \int_0^T \left\{ \frac{1}{2} \left[\langle y(t), a_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H + \langle Y(t), b_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H \right] \right. \\ & \quad \left. + \chi_{E_\varepsilon}(t) \left[\langle y(t), \delta a(t) \rangle_H + \langle Y, \delta b_1(t)x_2^\varepsilon(t) \rangle_H \right] \right\} dt. \end{aligned} \quad (9.59)$$

According to (9.57)–(9.59), we conclude that

$$\begin{aligned} & \mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\ &= \frac{1}{2} \text{Re} \mathbb{E} \int_0^T \left[\langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H - \langle y(t), a_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H \right. \\ & \quad \left. - \langle Y, b_{11}(t)(x_2^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H \right] dt + \text{Re} \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \left[\delta g(t) - \langle y(t), \delta a(t) \rangle_H \right. \\ & \quad \left. - \langle Y(t), \delta b(t) \rangle_H \right] dt + \frac{1}{2} \text{Re} \mathbb{E} \langle h_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H + o(\varepsilon). \end{aligned} \quad (9.60)$$

Next, by the definition of the relaxed transposition solution to (1.10) (with P_T , $J(\cdot)$, $K(\cdot)$ and $F(\cdot)$ given by (9.3)), and noting (9.25), we obtain that

$$\begin{aligned} & -\mathbb{E}\langle h_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H + \mathbb{E} \int_0^T \langle \mathbb{H}_{xx}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t))x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H dt \\ &= \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle b_1(t)x_2^\varepsilon(t), P(t)^* \delta b(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), b_1(t)x_2^\varepsilon(t) \rangle_H dt \\ & \quad + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), \delta b(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt \\ & \quad + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle Q^{(0)}(0, 0, \delta b)(t), \delta b(t) \rangle_H dt. \end{aligned} \quad (9.61)$$

Now, we estimate the terms in the right hand side of (9.61). It is clear that $P(t)^* = P(t)$ for $t \in (0, T)$, and hence

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle b_1(t)x_2^\varepsilon(t), P(t)^* \delta b(t) \rangle_H dt \right| \\ & \leq |x_2^\varepsilon(\cdot)|_{L_{\mathbb{F}}^\infty(0, T; L^4(\Omega; H))} \|b_1\|_{L_{\mathbb{F}}^\infty(0, T; \mathcal{L}(H))} \int_{E_\varepsilon} |P(t)^* \delta b(t)|_{L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega; H)} dt \\ & \leq C(x_0) \sqrt{\varepsilon} \int_{E_\varepsilon} |P(t) \delta b(t)|_{L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega; H)} dt = o(\varepsilon). \end{aligned} \quad (9.62)$$

Similarly,

$$\left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), b_1(t)x_2^\varepsilon(t) \rangle_H dt \right| = o(\varepsilon). \quad (9.63)$$

In what follows, for any $\tau \in [0, T]$, we choose $E_\varepsilon = [\tau, \tau + \varepsilon] \subset [0, T]$. We find a sequence $\{\beta_n\}_{n=1}^\infty \subset \mathcal{H}$ (recall (7.20) for the definition of \mathcal{H}) such that

$$\lim_{n \rightarrow \infty} \beta_n = \delta b \quad \text{in } L^4_{\mathbb{F}}(0, T; H).$$

Hence,

$$|\beta_n|_{L^4_{\mathbb{F}}(0, T; H)} \leq C(x_0) < \infty, \quad \forall n \in \mathbb{N}, \quad (9.64)$$

and there is a subsequence $\{n_k\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty$ such that

$$\lim_{k \rightarrow \infty} |\beta_{n_k}(t) - \delta b(t)|_{L^4_{\mathcal{F}_t}(\Omega; H)} = 0 \quad \text{for a.e. } t \in [0, T]. \quad (9.65)$$

For any $\ell \in \mathbb{N}$, let $t_j = \frac{j-1}{\ell}T$ for $j = 1, \dots, \ell + 1$. Since the set of simple processes is dense in $L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))$, we can find a $b_1^\ell \equiv b_1^\ell(t, \omega) = \sum_{j=1}^\ell \chi_{[t_j, t_{j+1})}(t) f_j(\omega)$, where $f_j \in L^\infty_{\mathcal{F}_{t_j}}(\Omega; \mathcal{L}(D(A)))$, such that

$$\lim_{\ell \rightarrow \infty} |b_1^\ell - b_1|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))} = 0. \quad (9.66)$$

It follows that

$$|b_1^\ell|_{L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))} \leq C(x_0) < \infty, \quad \forall \ell \in \mathbb{N}. \quad (9.67)$$

Denote by $(P^\ell(\cdot), Q^{(\cdot, \ell)}, \widehat{Q}^{(\cdot, \ell)})$ the relaxed transposition solution to the equation (1.10) with K replaced by b_1^ℓ , and P_T , J and F given as in (9.3). Also, denote by Q^ℓ and \widehat{Q}^ℓ the corresponding pointwisely defined linear operators from \mathcal{H} to $L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))$, given in Theorem 7.2. By Theorem 7.1 and noting (9.66)–(9.67), we see that

$$\begin{cases} \lim_{\ell \rightarrow \infty} \|Q^{(0, \ell)}(0, 0, \cdot) - Q^{(0)}(0, 0, \cdot)\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} = 0, \\ \lim_{\ell \rightarrow \infty} \|\widehat{Q}^{(0, \ell)}(0, 0, \cdot) - \widehat{Q}^{(0)}(0, 0, \cdot)\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} = 0. \end{cases} \quad (9.68)$$

Consider the following equation:

$$\begin{cases} dx_{2, n_k}^{\varepsilon, \ell} = [Ax_{2, n_k}^{\varepsilon, \ell} + a_1(t)x_{2, n_k}^{\varepsilon, \ell}]dt + [b_1^\ell(t)x_{2, n_k}^{\varepsilon, \ell} + \chi_{E_\varepsilon}(t)\beta_{n_k}(t)]dw(t) & \text{in } (0, T], \\ x_{2, n_k}^{\varepsilon, \ell}(0) = 0. \end{cases} \quad (9.69)$$

We have

$$\begin{aligned} & \mathbb{E}|x_{2, n_k}^{\varepsilon, \ell}(t)|_H^4 \\ &= \mathbb{E} \left| \int_0^t S(t-s)a_1(s)x_{2, n_k}^{\varepsilon, \ell}(s)ds + \int_0^t S(t-s)b_1^\ell(s)x_{2, n_k}^{\varepsilon, \ell}(s)dw(s) \right. \\ & \quad \left. + \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\beta_{n_k}(s)dw(s) \right|_H^4 \\ &\leq C \left[\mathbb{E} \left| \int_0^t S(t-s)a_1(s)x_{2, n_k}^{\varepsilon, \ell}(s)ds \right|_H^4 + \mathbb{E} \left| \int_0^t S(t-s)b_1^\ell(s)x_{2, n_k}^{\varepsilon, \ell}(s)dw(s) \right|_H^4 \right. \\ & \quad \left. + \mathbb{E} \left| \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\beta_{n_k}(s)dw(s) \right|_H^4 \right] \\ &\leq C \left[\int_0^t |b_1^\ell(s)|_{L^\infty(\Omega; \mathcal{L}(H))}^4 \mathbb{E}|x_{2, n_k}^{\varepsilon, \ell}(s)|_H^4 ds + \varepsilon \int_{E_\varepsilon} \mathbb{E}|\beta_{n_k}(s)|_H^4 ds \right]. \end{aligned} \quad (9.70)$$

By (9.64) and (9.67), thanks to Gronwall's inequality, (9.70) leads to

$$|x_{2,n_k}^{\varepsilon,\ell}(\cdot)|_{L_{\mathbb{F}}^{\infty}(0,T;L^4(\Omega;H))}^4 \leq C(x_0, \ell, k)\varepsilon^2. \quad (9.71)$$

Here and henceforth, $C(x_0, \ell, k)$ is a generic constant (depending on x_0, ℓ, k, T, A and C_L), which may be different from line to line. For any fixed $i, k \in \mathbb{N}$, since $Q^\ell \beta_{n_k} \in L_{\mathbb{F}}^2(0, T; L^{\frac{4}{3}}(\Omega; H))$, by (9.71), we find that

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle (Q^\ell \beta_{n_k})(t), x_{2,n_k}^{\varepsilon,\ell}(t) \rangle_H dt \right| \leq |x_{2,n_k}^{\varepsilon,\ell}(\cdot)|_{L_{\mathbb{F}}^{\infty}(0,T;L^4(\Omega;H))} \int_{E_\varepsilon} |(Q^\ell \beta_{n_k})(t)|_{L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega;H)} dt \\ & \leq C(x_0, \ell, k) \sqrt{\varepsilon} \int_{E_\varepsilon} |(Q^\ell \beta_{n_k})(t)|_{L_{\mathcal{F}_t}^{\frac{4}{3}}(\Omega;H)} dt = o(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (9.72)$$

Similarly,

$$\left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle x_{2,n_k}^{\varepsilon,\ell}(t), (\widehat{Q}^\ell \beta_{n_k})(t) \rangle_H dt \right| = o(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0. \quad (9.73)$$

From (7.21) in Theorem 7.2, and noting that both Q^ℓ and \widehat{Q}^ℓ are pointwisely defined, we arrive at the following equality:

$$\begin{aligned} & \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0,\ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt + \mathbb{E} \int_0^T \langle Q^{(0,\ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t), \chi_{E_\varepsilon} \beta_{n_k}(t) \rangle_H dt \\ & = \mathbb{E} \int_0^T \chi_{E_\varepsilon} \left[\langle (Q^\ell \beta_{n_k})(t), x_{2,n_k}^{\varepsilon,\ell}(t) \rangle_H + \langle x_{2,n_k}^{\varepsilon,\ell}(t), (\widehat{Q}^\ell \beta_{n_k})(t) \rangle_H \right] dt. \end{aligned} \quad (9.74)$$

Hence,

$$\begin{aligned} & \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt + \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t), \chi_{E_\varepsilon}(t) \delta b(t) \rangle_H dt \\ & - \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \left[\langle (Q^\ell \beta_{n_k})(t), x_{2,n_k}^{\varepsilon,\ell}(t) \rangle_H + \langle x_{2,n_k}^{\varepsilon,\ell}(t), (\widehat{Q}^\ell \beta_{n_k})(t) \rangle_H \right] dt \\ & = \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt + \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t), \chi_{E_\varepsilon}(t) \delta b(t) \rangle_H dt \\ & - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0,\ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \\ & - \mathbb{E} \int_0^T \langle Q^{(0,\ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t), \chi_{E_\varepsilon}(t) \beta_{n_k}(t) \rangle_H dt. \end{aligned} \quad (9.75)$$

It is easy to see that

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0,\ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| \\ & \leq \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| \\ & + \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| \\ & + \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0,\ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right|. \end{aligned} \quad (9.76)$$

From (9.65) and the density of the Lebesgue point, we find that for a.e. $\tau \in [0, T]$, it holds that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t), \chi_{E_\varepsilon}(t) \delta b(t) \rangle_H dt \right. \\
& \quad \left. - \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t), \chi_{E_\varepsilon}(t) \beta_{n_k}(t) \rangle_H dt \right| \\
& \leq \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)|_{L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \\
& \leq C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |\chi_{E_\varepsilon} \delta b|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \\
& \leq C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{|\delta b(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)}}{\sqrt{\varepsilon}} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \\
& = C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} |\delta b(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)} \left[\frac{1}{\varepsilon} \int_\tau^{\tau+\varepsilon} |\delta b(t) - \beta_{n_k}(t)|_{L^4_{\mathcal{F}_t}(\Omega; H)}^2 dt \right]^{\frac{1}{2}} \\
& = C \lim_{k \rightarrow \infty} |\delta b(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)} |\delta b(\tau) - \beta_{n_k}(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)} \\
& = 0.
\end{aligned} \tag{9.77}$$

Similarly,

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right. \\
& \quad \left. - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| \\
& \leq \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |\widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})|_{L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \\
& \leq C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} |\chi_{E_\varepsilon} \beta_{n_k}|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \\
& \leq C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left\{ |\chi_{E_\varepsilon} \delta b|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \right. \\
& \quad \left. + \int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right\} \\
& \leq C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \left\{ \frac{|\delta b(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)}}{\sqrt{\varepsilon}} \left[\int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right]^{\frac{1}{2}} \right. \\
& \quad \left. + \frac{1}{\varepsilon} \int_0^T \chi_{E_\varepsilon}(t) \left(\mathbb{E} |\delta b(t) - \beta_{n_k}(t)|_H^4 \right)^{\frac{1}{2}} dt \right\} \\
& = C \lim_{k \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \left\{ |\delta b(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)} \left[\frac{1}{\varepsilon} \int_\tau^{\tau+\varepsilon} |\delta b(t) - \beta_{n_k}(t)|_{L^4_{\mathcal{F}_t}(\Omega; H)}^2 dt \right]^{\frac{1}{2}} \right. \\
& \quad \left. + \frac{1}{\varepsilon} \int_\tau^{\tau+\varepsilon} |\delta b(t) - \beta_{n_k}(t)|_{L^4_{\mathcal{F}_t}(\Omega; H)}^2 dt \right\} \\
& = C \lim_{k \rightarrow \infty} [|\delta b(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)} |\delta b(\tau) - \beta_{n_k}(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)} + |\delta b(\tau) - \beta_{n_k}(\tau)|_{L^4_{\mathcal{F}_\tau}(\Omega; H)}] \\
& = 0.
\end{aligned} \tag{9.78}$$

From (9.68) and the density of the Lebesgue point, we find that for a.e. $\tau \in [0, T]$, it holds that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right. \\
& \quad \left. - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0, \ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| \\
& \leq \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \chi_{E_\varepsilon} \beta_{n_k} \right|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))} \left(\int_0^T \left| \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k}) \right. \right. \\
& \quad \left. \left. - \widehat{Q}^{(0, \ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k}) \right|_{L^{\frac{4}{3}}_{\mathcal{F}_s}(\Omega; H)} ds \right)^{\frac{1}{2}} \\
& \leq \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \chi_{E_\varepsilon} \beta_{n_k} \right|_{L^2_{\mathbb{F}}(0, T; L^4(\Omega; H))}^2 \left\| \widehat{Q}^{(0, \ell)}(0, 0, \cdot) - \widehat{Q}^{(0)}(0, 0, \cdot) \right\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\
& = \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \left| \beta_{n_k}(\tau, \cdot) \right|_{L^4(\Omega; H)}^2 \left\| \widehat{Q}^{(0, \ell)}(0, 0, \cdot) - \widehat{Q}^{(0)}(0, 0, \cdot) \right\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\
& = 0.
\end{aligned} \tag{9.79}$$

From (9.76)–(9.79), we find that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt \right. \\
& \quad \left. - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \widehat{Q}^{(0, \ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| = 0.
\end{aligned} \tag{9.80}$$

By a similar argument, we obtain that

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t), \chi_{E_\varepsilon}(t) \delta b(t) \rangle_H dt \right. \\
& \quad \left. - \mathbb{E} \int_0^T \langle Q^{(0, \ell)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t), \chi_{E_\varepsilon}(t) \beta_{n_k}(t) \rangle_H dt \right| = 0.
\end{aligned} \tag{9.81}$$

From (9.72)–(9.73), (9.74)–(9.75) and (9.80)–(9.81), we obtain that

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle \delta b(t), \widehat{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle Q^{(0)}(0, 0, \delta b)(t), \delta b(t) \rangle_H dt \right| \\
& = o(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned} \tag{9.82}$$

Therefore, we have

$$\begin{aligned}
& \mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \\
& = \text{Re} \mathbb{E} \int_0^T \left[\delta g(t) - \langle y(t), \delta a(t) \rangle_H - \langle Y(t), \delta b(t) \rangle_H - \frac{1}{2} \langle P(t) \delta b(t), \delta b(t) \rangle_H \right] \chi_{E_\varepsilon}(t) dt + o(\varepsilon).
\end{aligned} \tag{9.83}$$

Since $\bar{u}(\cdot)$ is the optimal control, $\mathcal{J}(u^\varepsilon(\cdot)) - \mathcal{J}(\bar{u}(\cdot)) \geq 0$. Thus,

$$\text{Re} \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \left[\langle y(t), \delta a(t) \rangle_H + \langle Y(t), \delta b(t) \rangle_H - \delta g(t) + \frac{1}{2} \langle P(t) \delta b(t), \delta b(t) \rangle_H \right] dt \leq o(\varepsilon), \tag{9.84}$$

as $\varepsilon \rightarrow 0$.

Finally, similar to [14, 30], from (9.84), we obtain (9.4). This completes the proof of Theorem 9.1. \square

Remark 9.1 1) We believe that $b_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))$ is a technical condition in Theorem 9.1 but we cannot drop it at this moment (because we need to use Theorem 7.2). It is easy to see that this condition is satisfied for one of the following cases:

- i) The operator A is a bounded linear operator on H ;
- ii) The diffusion term $b(t, x, u)$ is independent of the state variable x ; or
- iii) Some further regularities for x_0 , $a(\cdot, \cdot, \cdot)$ and $b(\cdot, \cdot, \cdot)$ are imposed, say $x_0 \in L^8_{\mathcal{F}_0}(\Omega; D(A))$, and the Assumption (A1) holds also when the space H is replaced by $D(A)$.

2) If the equation (1.10), with P_T , $J(\cdot)$, $K(\cdot)$ and $F(\cdot)$ given by (9.3), admits a transposition solution $(P(\cdot), Q(\cdot))$, then the assumption $b_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))$ is not needed (for the same conclusion in Theorem 9.1). Indeed, in this case, by Definition 1.2, instead of (9.61), we have

$$\begin{aligned}
& -\mathbb{E}\langle h_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H + \mathbb{E} \int_0^T \langle \mathbb{H}_{xx}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t))x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H dt \\
& = \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle b_1(s)x_2^\varepsilon(s), P(s)^* \delta b(s) \rangle_H ds + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle P(s) \delta b(s), b_1(s)x_2^\varepsilon(s) \rangle_H ds \\
& \quad + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle P(s) \delta b(s), \delta b(s) \rangle_H ds + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle Q(s) \delta b(s), x_2^\varepsilon(s) \rangle_H ds \\
& \quad + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle Q(s)x_2^\varepsilon(s), \delta b(s) \rangle_H ds.
\end{aligned} \tag{9.85}$$

The estimates (9.62)-(9.63) are still valid. On the other hand, by $Q(\cdot)\delta b(\cdot) \in L^1_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H))$, it holds that

$$\begin{aligned}
& \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle Q(s) \delta b(s), x_2^\varepsilon(s) \rangle_H ds \\
& \leq |x_2^\varepsilon(s)|_{L^\infty(0, T; L^4(\Omega; H))} \int_{E_\varepsilon} |Q(s) \delta b(s)|_{L^{\frac{4}{3}}(\Omega; H)} ds \\
& \leq C\sqrt{\varepsilon} \int_{E_\varepsilon} |Q(s) \delta b(s)|_{L^{\frac{4}{3}}(\Omega; H)} ds = o(\varepsilon).
\end{aligned} \tag{9.86}$$

Similarly, noting that $Q(t)^* = Q(t)$ for a.e. $t \in (0, T)$, we obtain that

$$\left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle Q(s)x_2^\varepsilon(s), \delta b(s) \rangle_H ds \right| = \left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(s) \langle x_2^\varepsilon(s), Q(s)^* \delta b(s) \rangle_H ds \right| = o(\varepsilon). \tag{9.87}$$

Combining (9.85), (9.62)-(9.63) and (9.86)-(9.87), we still have (9.83), which leads to the desired result.

3) For concrete equations, say for the controlled stochastic heat equations, one may obtain better results than that of Theorem 9.1. Related work will be presented elsewhere.

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